From the Hyperboloid to the Disk: Models, Geodesics, and Isometries of the Hyperbolic Plane

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FROM THE HYPERBOLOID TO THE DISK: MODELS, GEODESICS, AND ISOMETRIES OF THE HYPERBOLIC PLANE

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Abstract. Hyperbolic geometry is a non-Euclidean geometry where geodesics play the role of straight lines. It is characterised by a constant negative curvature. The idea was conceived when mathematicians tried to understand the parallel postulate. This expository paper develops the foundations of the hyperbolic plane using the classical models, such as the hyperboloid, Beltrami–Klein, Poincaré disk, and upper half-plane models. It examines Gaussian curvature, hyperbolic distance, geodesics, metric transformations, Möbius transformations, and isometry groups, and explains how the different models represent the same geometry. The paper also discusses hyperbolic tessellations and the applications of Hyperbolic Geometry in mathematics, physics, art, and data representation.

1. Introduction

Euclidean geometry dictated mathematical thinking for two millennia with its five postulates. For centuries, mathematicians tried to prove the parallel postulate using the first four but failed. Gauss, Bolyai, and Lobachevsky negated the fifth postulate, and a new geometry stemmed from the failure of Euclid's fifth postulate known as **Hyperbolic Geometry**. Gauss made the remarkable discovery that denying *fifth postulate* could lead to a geometry called the *Non-Euclidean Geometry.* He recognised that assuming the sum of the angles of a triangle to be less than 180° is equivalent to rejecting Euclid's fifth postulate. Bolyai, and Lobachevsky independently published papers on Non-Euclidean Geometry providing strong evidence for its consistency, highlighting duality between non-Euclidean and spherical trigonometry where the hyperbolic trigonometric functions used in non-Euclidean geometry play a role analogous to that of ordinary trigonometric functions in Euclidean geometry.

2. Origin of Hyperbolic Geometry

2.1. **Euclid's Postulates:** 1

(1) A straight line segment can be drawn by joining any two points.
(2) Any straight line segment can be extended indefinitely in a straight line.
(3) Given any straight-line segment, a circle can be drawn, with the segment as a radius and one endpoint as the center.
(4) All right angles are congruent.
(5) If two lines are drawn which intersect a third in such a way that the sum of the inner angles on one side is less than two right angles, then the two lines inevitably must intersect each other on that side if extended far enough. This postulate is equivalent to what is known as the parallel postulate.

For centuries, scientists tried to prove the fifth postulate or the **parallel postulate** from the first four, but were forced to make additional assumptions to conclude it. During this period, Gauss, Lobachevsky, and Bolyai, independently developed a new type of geometry, now known as the Hyperbolic Geometry, which directly contradicted the

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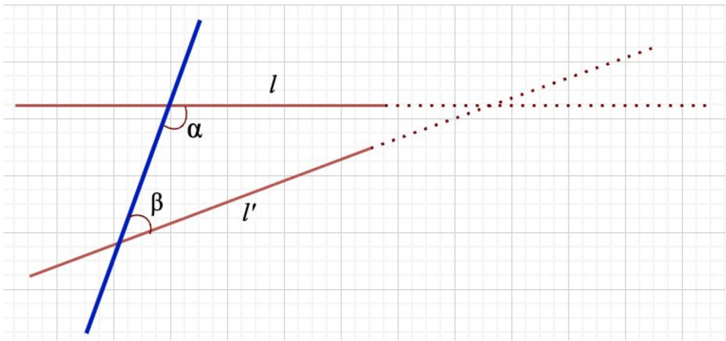

FIGURE 1. The fifth postulate: If $\alpha + \beta < 2R$, $(R = 90°)$ then ℓ and ℓ' intersect on the right side

parallel postulate of Euclidean geometry. **Lobachevsky** introduced the concept of the **angle of parallelism**: for a line l and a point A, a perpendicular distance a from it, the angle of parallelism α is the smallest angle such that the line l drawn from A, remains parallel to l, that is, it does not intersect l.

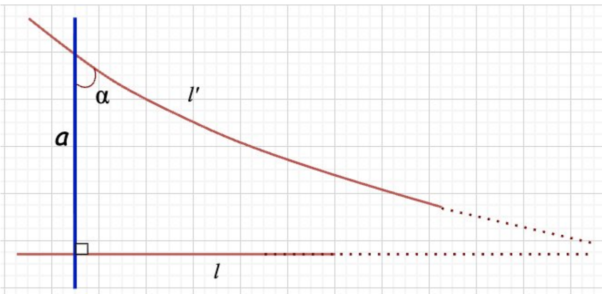

FIGURE 2. Angle of Parallelism

2.2. **Spherical-Hyperbolic Contrasts:** Spherical and hyperbolic geometry are opposite in many ways, starting from the way they handle the parallel postulate. In spherical geometry, the opposite of a straight line is **great circle**.
In spherical geometry, the parallel postulate takes the form of
"Through a point not on a given line, there is no line parallel to the given line".
In hyperbolic geometry, the parallel postulate states that
"Through a point not on a given line, there are infinitely many lines parallel to the given line".
The duality between spherical and hyperbolic geometry can be further explained by highlighting the contrasting properties of a triangle in each space.

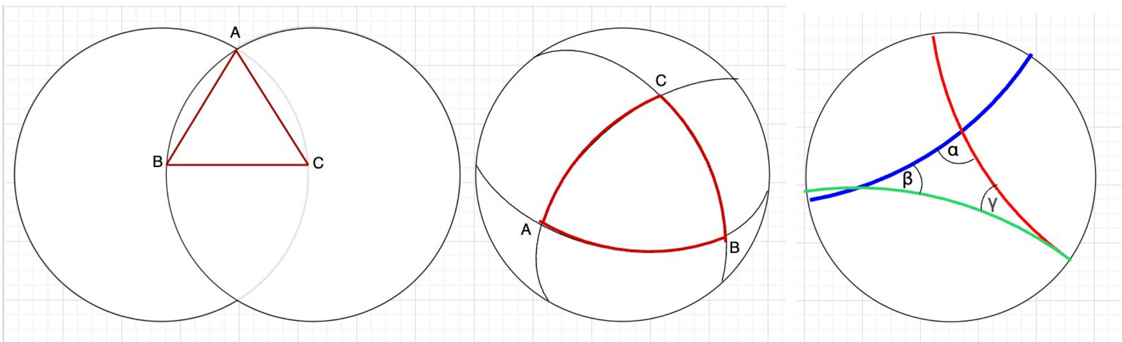

FIGURE 3. Euclidean Triangle, Spherical Triangle and Hyperbolic Triangle

- In **Euclidean geometry**, the angle sum is exactly $\alpha + \beta + \gamma = 180°$.
- In **spherical geometry**, the angle sum of a triangle satisfies $\alpha + \beta + \gamma > 180°$.
- In **hyperbolic geometry**, the angle sum obeys $\alpha + \beta + \gamma < 180°$.

TABLE 1. Comparison of Spherical, Euclidean, and Hyperbolic Geometry

Property	**Spherical Geometry**	**Euclidean Geometry**	**Hyperbolic Geometry**
Lines	Great circles that eventually converge	Infinite straight lines that never meet	Geodesics; model-dependent Euclidean appearance
Sum of angles in a triangle	$\alpha + \beta + \gamma > 180°$	$\alpha + \beta + \gamma = 180°$	$\alpha + \beta + \gamma < 180°$
Number of parallels through a point not on a line	0	1	Infinite
Circumference of a circle	$2\pi \sin r$	$2\pi r$	$2\pi \sinh r$
Area of a circle	$2\pi(1 - \cos r)$	πr^2	$2\pi(\cosh r - 1)$
Pythagorean formula	$\cos c = \cos a \cos b$ (for right triangle)	$a^2 + b^2 = c^2$	$\cosh c = \cosh a \cosh b$ (for right triangle)
Area of triangle	$k((\alpha + \beta + \gamma) - \pi)$	Not proportional to angles	$k(\pi - (\alpha + \beta + \gamma))$
Parallel postulate	Does not hold (no parallels)	Holds	Does not hold (multiple parallels)
Force analogy	Squishing effect (like on a sphere)	No distortion	Tidal effect (stretching in opposite directions)
Tiling of space	Only certain polygons tile the surface	Regular tilings like squares, triangles, hexagons	Many tilings possible, even with heptagons, octagons etc.
Geodesics	Great circles	Straight lines	Arcs, semicircles, or rays (depending on model)
Model examples	Surface of Earth, $\mathbb{S}^2$	$\mathbb{R}^2$, the flat plane	Poincaré disk, half-plane, hyperboloid

3. Curvature

3.1. **What is Curvature?** The fundamental geometric property of a space that establishes the relationship between the angles of a triangle and its area is called **curvature**. It also establishes the relationship between the circumference of a circle and its radius. Curvature measures the extent to which the space is curved and in which direction it is curved. Consider a smooth curve. The curvature at any point of the curve quantifies how sharply it bends away from the tangent line. The *radius of curvature* at that point refers to the radius of the best-fitting circle called the *osculating circle* that approximates the circle locally. The curvature of the circle is the reciprocal of the radius and is given by:

$$\kappa = \frac{1}{r}$$

For surfaces, curvature measures how much the surface deviates from the tangent plane at a given point. For surfaces, Gaussian curvature is denoted by K. Surfaces of constant Gaussian curvature are classified according to the sign of K:

- **Positive curvature** ($K > 0$)**:** The surface lies entirely on one side of the tangent plane at any point (e.g. sphere). If the surface is simply connected, it is a sphere with curvature.
- **Negative curvature** ($K < 0$)**:** The surface is saddle-shaped and the tangent plane intersects the surface on both sides (e.g. a hyperbolic paraboloid). A simply connected surface with negative curvature is the hyperbolic plane (or 2-dimensional hyperbolic space). To be more precise, the hyperbolic plane is the unique simply connected and complete surface with constant Gaussian curvature $K = -1$. While there can exist surfaces with negative curvature that are not the hyperbolic plane, the curvature should be constant and equal to -1. A saddle-shaped surface is a helpful visualisation of negative curvature, but it is not necessarily a model of the hyperbolic plane.
- **Zero curvature** ($K = 0$)**:** The surface is flat in at least one direction and the straight line lies in the tangent plane at every point (e.g. a cylinder). A simply connected surface with zero curvature is the Euclidean plane.

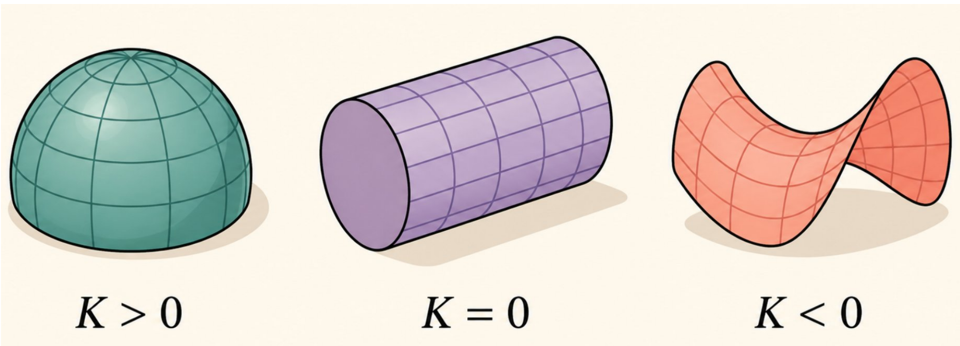

Figure 4. Interpreting the Gaussian curvature's value

Verification of $K = -1$ in the upper half-plane: The statement that the hyperbolic plane has constant curvature $K = -1$ can be checked directly in the upper half-plane model. The metric is

$$ds^2 = \frac{dx^2 + dy^2}{y^2}.$$

This is a conformal metric of the form

$$ds^2 = \lambda^2(dx^2 + dy^2),$$

where
$$\lambda = \frac{1}{y}.$$
For a conformal metric, the Gaussian curvature is given by
$$K = -\frac{\Delta(\log \lambda)}{\lambda^2},$$
where
$$\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$$
is the Euclidean Laplacian. Since
$$\log \lambda = \log\left(\frac{1}{y}\right) = -\log y,$$
we compute
$$\frac{\partial^2}{\partial x^2}(\log \lambda) = 0$$
and
$$\frac{\partial}{\partial y}(-\log y) = -\frac{1}{y}, \qquad \frac{\partial^2}{\partial y^2}(-\log y) = \frac{1}{y^2}.$$
Therefore,
$$\Delta(\log \lambda) = \frac{1}{y^2}.$$
Substituting into the curvature formula gives
$$K = -\frac{\Delta(\log \lambda)}{\lambda^2} = -\frac{\frac{1}{y^2}}{\frac{1}{y^2}} = -1.$$
Thus the upper half-plane model has constant Gaussian curvature $K = -1$, which is the defining curvature of the hyperbolic plane.

Theorem 3.1 (Uniformisation theorem for closed orientable surfaces)**.** *Every closed, connected, orientable surface admits a Riemannian metric of constant Gaussian curvature.*

According to its topology, its universal covering space is, after an appropriate rescaling, isometric to exactly one of
$$\mathbb{S}^2, \qquad \mathbb{R}^2, \qquad \mathbb{H}^2.$$
Consequently, the surface may be represented as a quotient
$$S \cong M/\Gamma,$$
where M is one of the three simply connected model spaces above and Γ is a discrete group of isometries acting properly discontinuously on M.

3.2. **Geometric Interpretation of Curvature:** Take a unit speed curve $\gamma(t)$ in $\mathbb{R}^2$. When we move from t to $t + \Delta t$, the curve deviates from its tangent line at $\gamma(t)$. The perpendicular distance from a point on the curve to this tangent line is given by
$$(\gamma(t + \Delta t) - \gamma(t)) \cdot \mathbf{n},$$
where $\mathbf{n}$ is a unit vector perpendicular to the tangent vector $\dot{\gamma}(t)$ at the point $\gamma(t)$. By Taylor expansion, we have:
$$\gamma(t + \Delta t) = \gamma(t) + \dot{\gamma}(t)\Delta t + \frac{1}{2}\ddot{\gamma}(t)(\Delta t)^2 + O((\Delta t)^3).$$

Since $\dot{\gamma}(t) \cdot \mathbf{n} = 0$, because $\mathbf{n}$ is normal to the tangent vector, the first-order term vanishes when projected onto $\mathbf{n}$. Therefore, for small Δt, we obtain

$$\Big(\gamma(t+\Delta t) - \gamma(t)\Big) \cdot \mathbf{n} \approx \frac{1}{2}\ddot{\gamma}(t) \cdot \mathbf{n}\,(\Delta t)^2.$$

This shows that the component of the second derivative in the direction normal to the curve gives a measure of how much the curve deviates from its tangent line.

Definition 3.1. *Let $\gamma(t) : I \to \mathbb{R}^3$ be a curve parametrized by arc length $s \in I$. The number $|\ddot{\gamma}(t)| = \kappa(s)$ is called the curvature of γ at s.*

3.3. **Key properties of surfaces with Negative Curvature:**

- **Local saddle-shape behavior:** A surface with negative Gaussian curvature is locally non-convex and lies on both sides of tangent plane, exhibiting saddle-like geometry. In other words, a surface with **negative Gaussian Curvature** bends in opposite directions at a point (like a saddle) making the surface **non-convex** and lies partly above and below its tangent plane.
- **Convexity of distance and thin-triangle behaviour:** In a complete, simply connected Riemannian manifold of nonpositive sectional curvature, the CAT(0) midpoint inequality holds. If m is the midpoint of the geodesic segment joining x and y, then
$$d(z,m)^2 \le \frac{1}{2}d(z,x)^2 + \frac{1}{2}d(z,y)^2 - \frac{1}{4}d(x,y)^2.$$
For spaces whose curvature is bounded above by a negative constant, geodesic triangles also satisfy stronger hyperbolic thinness properties.

4. Classical Models of Hyperbolic Geometry

4.1. **Hyperboloid Model.** The hyperbolic model, also called the Minkowski model, is an n-dimensional model in which each point in the hyperbolic space is shown as a point on the upper sheet of the curved surface called a two-sheeted hyperboloid.

Definition 4.1. *Let*

$$\mathbb{H} = \Big\{(x,y,z) \in \mathbb{R}^3 \;\Big|\; x^2+y^2-z^2 = -1\Big\} = \Big\{p \in \mathbb{R}^{2,1} \;\Big|\; \langle p,p\rangle = -1\Big\} \subset \mathbb{R}^{2,1}$$

be the two-sheeted hyperboloid in Minkowski space $\mathbb{R}^{2,1}$. Then the upper sheet of the hyperboloid is defined as

$$H^+ = H \cap \{z > 0\}.$$

Metric: The ambient Minkowski space $\mathbb{R}^{2,1}$ is equipped with the Lorentzian inner product

$$\langle u,v\rangle_L = u_1v_1 + u_2v_2 - u_0v_0.$$

Equivalently, the ambient line element is

$$ds_L^2 = -dx_0^2 + dx_1^2 + dx_2^2.$$

Although this form is indefinite on the ambient space, its restriction to the tangent plane

$$T_pH^+ = \{v \in \mathbb{R}^{2,1} : \langle p,v\rangle_L = 0\}$$

is positive definite. This restricted form defines the Riemannian metric of the hyperboloid model.

Distance Function: The distance between two points $\boldsymbol{p}, \boldsymbol{q} \in H^+$ is defined using the Lorentzian inner product: $d(\boldsymbol{p},\boldsymbol{q}) = \operatorname{arcosh}\left(-\langle \boldsymbol{p},\boldsymbol{q}\rangle\right)$

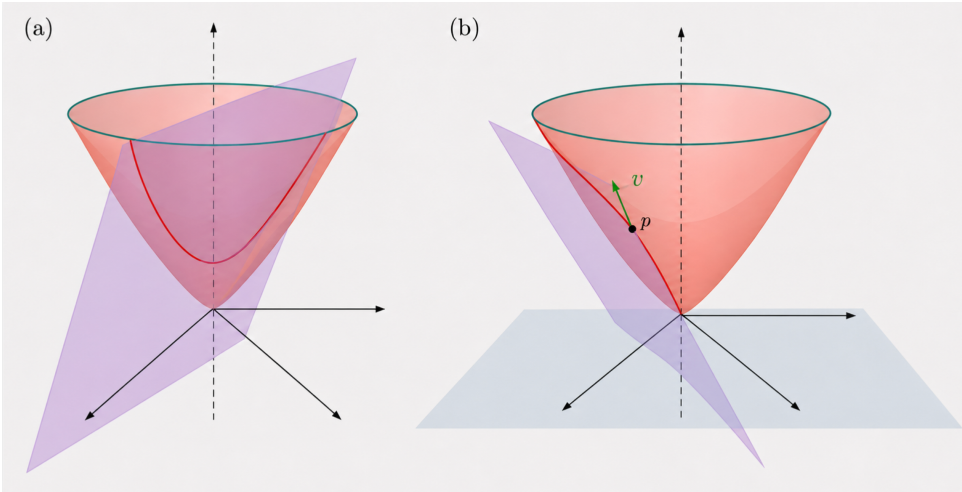

FIGURE 5. (a) The hyperboloid $\mathbb{H}$ in $\mathbb{R}^{n,1}$ (b) Tangent space to the hyperboloid at a point. Adapted from [4].

This formula expresses the hyperbolic distance in terms of the Lorentzian angle between the timelike unit vectors p and q.

$$d(\boldsymbol{p}, \boldsymbol{q}) = \operatorname{arcosh}\left(-\langle \boldsymbol{p}, \boldsymbol{q}\rangle_L\right).$$

Geodesics: Geodesics in the hyperboloid model are curves that locally minimise distance. They correspond to the intersection of the hyperboloid with planes through the origin:

$$\gamma = H^+ \cap P,$$

where P is a two-dimensional linear subspace of $\mathbb{R}^{2,1}$.

Let $\boldsymbol{p} \in H^+$ and let $\boldsymbol{v} \in T_{\boldsymbol{p}}H^+$ be a unit tangent vector satisfying

$$\langle \boldsymbol{p}, \boldsymbol{v}\rangle_L = 0, \qquad \langle \boldsymbol{v}, \boldsymbol{v}\rangle_L = 1.$$

Then the geodesic through $\boldsymbol{p}$ with initial velocity $\boldsymbol{v}$ is

$$\gamma(t) = \cosh t\, \boldsymbol{p} + \sinh t\, \boldsymbol{v}.$$

Indeed,

$$\langle \gamma(t), \gamma(t)\rangle_L = -\cosh^2 t + \sinh^2 t = -1,$$

so $\gamma(t) \in H^+$.

Isometries: The isometry group of the hyperboloid model is the subgroup of Lorentz transformations that preserve the hyperboloid and its upper sheet:

$$O^+(2,1),$$

i.e., the group of linear transformations preserving the Minkowski inner product and the condition $z > 0$. The orientation-preserving isometry group is the identity component

$$SO_0(2,1).$$

It is naturally isomorphic to the orientation-preserving isometry group of the upper half-plane:

$$\mathrm{Isom}^+(\mathbb{H}^2) \cong SO_0(2,1) \cong PSL(2,\mathbb{R}).$$

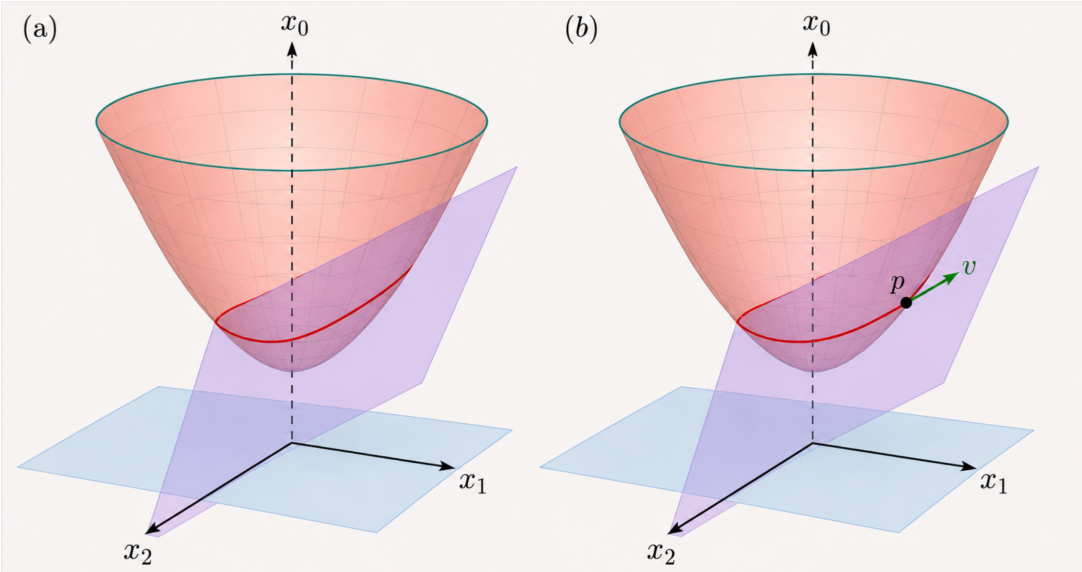

FIGURE 6. Geodesics on the hyperboloid model. Adapted from [4].

A matrix $A \in O^+(2,1)$ preserves the hyperboloid because it preserves the Lorentzian form

$$\langle x, x\rangle = -x_0^2 + x_1^2 + x_2^2.$$

Equivalently,

$$A^T J A = J, \qquad J = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

For example, the Lorentz boost

$$A_t = \begin{pmatrix} \cosh t & \sinh t & 0 \\ \sinh t & \cosh t & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

satisfies $A_t^T J A_t = J$. Hence it preserves the equation

$$-x_0^2 + x_1^2 + x_2^2 = -1$$

and maps the hyperboloid to itself.

From the Hyperboloid to the Poincaré Disk. The relation between the hyperboloid model and the Poincaré disk model can be expressed through stereographic projection. In the hyperboloid model, the hyperbolic plane is represented as the upper sheet of the double-sheeted hyperboloid

$$-x_0^2 + x_1^2 + x_2^2 = -1, \qquad x_0 \geq 1.$$

Equivalently, this is the sheet with $x_0 > 0$; the equation then automatically implies $x_0 \geq 1$, with equality only at $(1, 0, 0)$.

Project each point of this upper sheet from the pole $(-1, 0, 0)$, the bottom vertex of the lower sheet, onto the plane $x_0 = 0$. If $X = (x_0, x_1, x_2)$ is a point on the hyperboloid, then the line joining $(-1, 0, 0)$ to X meets the plane $x_0 = 0$ at

$$\left(0, \frac{x_1}{x_0 + 1}, \frac{x_2}{x_0 + 1}\right).$$

Thus the corresponding point in the Poincaré disk is

$$u = \left(\frac{x_1}{x_0+1}, \frac{x_2}{x_0+1} \right).$$

Since

$$|u|^2 = \frac{x_1^2 + x_2^2}{(x_0+1)^2} = \frac{x_0^2 - 1}{(x_0+1)^2} = \frac{x_0 - 1}{x_0 + 1} < 1$$

for $x_0 > 1$, while $x_0 = 1$ maps to the origin, the image lies inside the unit disk. Therefore, stereographic projection from $(-1, 0, 0)$ identifies the hyperboloid model with the Poincaré disk model: the same hyperbolic plane is represented inside a bounded Euclidean disk, whose boundary circle corresponds to points at infinity.

Under this stereographic projection, the Lorentzian metric induced on the hyperboloid becomes the Poincaré disk metric

$$ds^2 = \frac{4(du_1^2 + du_2^2)}{(1 - |u|^2)^2}.$$

Thus the projection does not merely place the hyperboloid inside the disk visually; it transfers the hyperbolic metric to the conformal disk metric.

Verification of the metric transfer. Let $u = (u_1, u_2)$ be the point in the Poincaré disk obtained from the hyperboloid by

$$u_1 = \frac{x_1}{x_0+1}, \qquad u_2 = \frac{x_2}{x_0+1}.$$

The inverse transformation is

$$x_0 = \frac{1 + |u|^2}{1 - |u|^2}, \qquad x_1 = \frac{2u_1}{1 - |u|^2}, \qquad x_2 = \frac{2u_2}{1 - |u|^2},$$

where $|u|^2 = u_1^2 + u_2^2$. A direct substitution shows that

$$-x_0^2 + x_1^2 + x_2^2 = -1,$$

so this inverse map indeed lands on the hyperboloid.

The Lorentzian metric on the ambient space is

$$ds^2 = -dx_0^2 + dx_1^2 + dx_2^2.$$

Substituting the expressions for x_0, x_1, x_2 in terms of u_1, u_2 and simplifying gives

$$-dx_0^2 + dx_1^2 + dx_2^2 = \frac{4(du_1^2 + du_2^2)}{(1 - |u|^2)^2}.$$

Therefore, under stereographic projection from the hyperboloid to the disk, the induced Lorentzian metric becomes exactly the Poincaré disk metric

$$ds^2 = \frac{4(du_1^2 + du_2^2)}{(1 - |u|^2)^2}.$$

This proves that the projection is not only a visual correspondence between models; it is a metric identification of the hyperboloid model with the Poincaré disk model.

4.2. **The Beltrami–Klein Model, also called the Cayley–Klein or Projective Model.** The Beltrami–Klein model, also known as the Cayley–Klein or projective model, represents the hyperbolic plane as the open unit disk

$$D = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 < 1\}.$$

These names refer to the same projective model of hyperbolic geometry; the different terminology reflects the historical contributions of Beltrami, Klein, and Cayley.

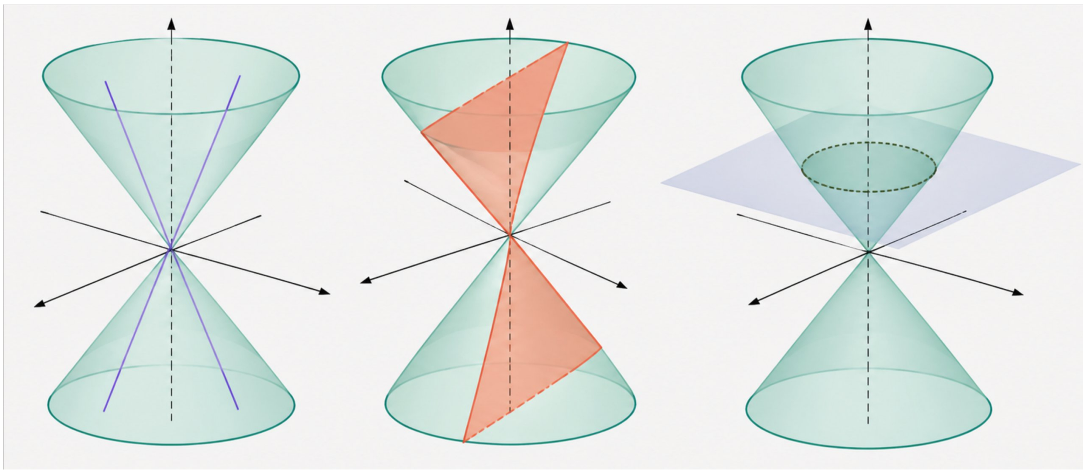

FIGURE 7. Cayley–Klein Model Adapted from [4].

This model can be obtained from the hyperboloid model by central projection onto the plane $x_0 = 1$. Unlike the Poincaré disk model, the Beltrami–Klein model is not conformal, so it does not preserve angles. Its main advantage is that geodesics appear as Euclidean straight-line chords inside the disk.

The metric on the Klein disk is

$$ds^2 = \frac{dx^2 + dy^2}{1 - x^2 - y^2} + \frac{(x\,dx + y\,dy)^2}{(1 - x^2 - y^2)^2}.$$

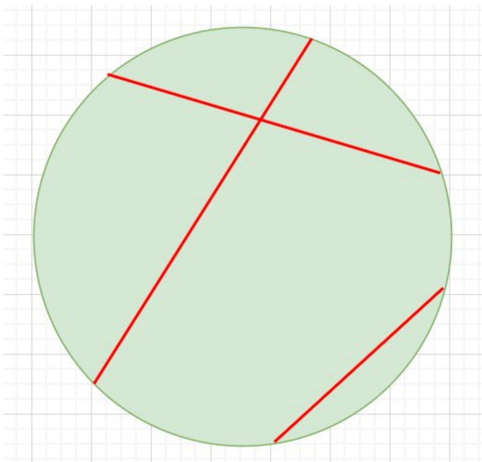

FIGURE 8. Beltrami-Klein Model Geodesics

The hyperbolic distance between two points $p, q \in D$ is

$$d(p,q) = \operatorname{arcosh}\left(\frac{1 - \langle p, q\rangle}{\sqrt{(1 - \|p\|^2)(1 - \|q\|^2)}}\right),$$

where $\langle p, q\rangle$ is the Euclidean dot product. Thus, the Beltrami–Klein model is especially useful for studying incidence, convexity, and geodesic structure, while the Poincaré disk is more useful for angle measurements.

4.3. **Poincaré Disk Model.**

Definition 4.2. *The Poincaré disk model or the conformal disk model represents the hyperbolic plane using the interior of the Euclidean unit circle:*

$$\mathbb{D} = \{(x, y) \in \mathbb{R}^2 \quad |x^2 + y^2 < 1\}$$

and equivalently, as a subset of the complex plane:

$$\mathbb{D} = \{z \in \mathbb{C} \mid |z| < 1\} \subset \mathbb{C}$$

Metric: The metric on the disk is given by:

$$ds^2 = \frac{4(dx^2 + dy^2)}{(1 - x^2 - y^2)^2} = \frac{4|dz|^2}{(1 - |z|^2)^2}$$

A hyperbolic line in the Poincaré disk model is either a Euclidean diameter of a unit circle or a circular arc that intersects the boundary circle at right angles. In the figure shown below, we have a hyperbolic line l and a point P such that many parallel lines to l pass through P.

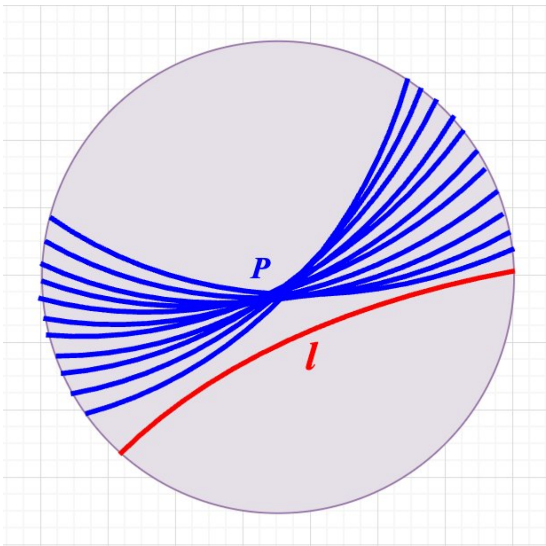

FIGURE 9. Parallel lines to hyperbolic line l and passing through point P

Lemma 4.1. *Every hyperbolic line in the Poincaré disk model is represented by one of the following:*

- ***Euclidean diameter*** *of the unit disk that passes through point $(c, d) \neq (0, 0)$, satisfying the Euclidean equation: $dx = cy$*
- *An **arc of a Euclidean circle** lying entirely within the unit disk and orthogonal to its boundary, given by the equation:*

$$x^2 + y^2 - 2ax - 2by + 1 = 0 \text{ where } a^2 + b^2 > 1$$

*The circle has **Center:** $C = (a, b)$ and **Radius:** $r = \sqrt{a^2 + b^2 - 1}$*

Distance Formula: The *hyperbolic distance* between two points P and Q in the Poincaré disk is defined as:

$$d(P, Q) := \cosh^{-1}\left(1 + \frac{2|PQ|^2}{(1 - |P|^2)(1 - |Q|^2)}\right)$$

where $|PQ|$ denotes the Euclidean distance between P and Q, and $|P|$, $|Q|$ denote the Euclidean distance of P and Q from the origin, respectively. Two **hyperbolic segments** are said to be *congruent* if they have the same hyperbolic length. The **angle** between

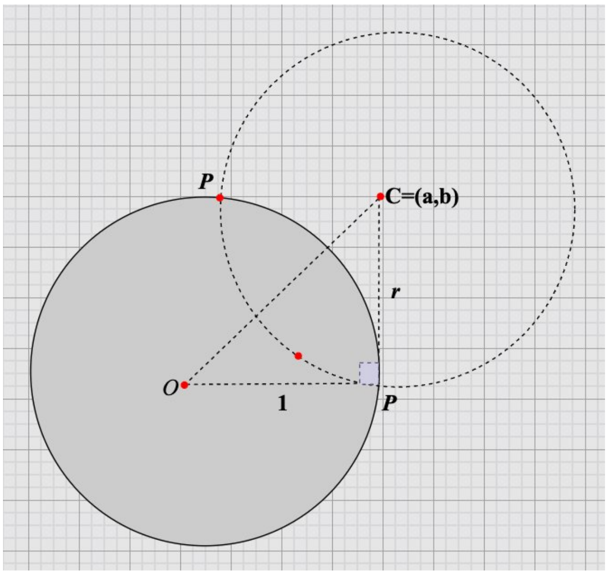

FIGURE 10. Hyperbolic line

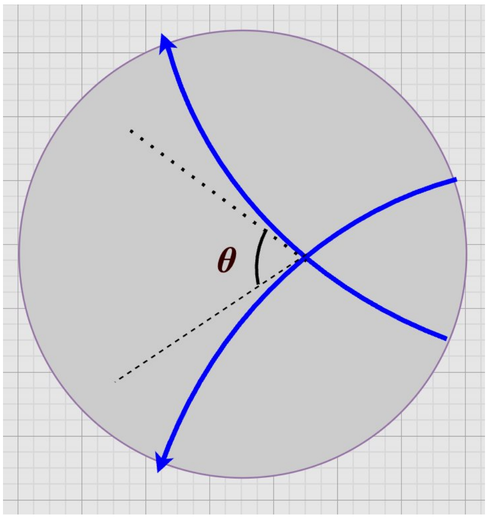

FIGURE 11. Hyperbolic distance

two hyperbolic rays is defined as the Euclidean angle between their tangent lines at the point of intersection. Angles are congruent if they have the same measure.

Lemma 4.2. *The* ***Hyperbolic distance*** *from a point P to the origin in Poincaré disk model is given by:*

$$d(O,P) = \cosh^{-1}\left(\frac{1+|P|^2}{1-|P|^2}\right) = \ln\left(\frac{1+|P|}{1-|P|}\right)$$

where $|P|$ denotes the Euclidean distance of the P from the origin.

Geodesics. The **geodesics** in the Poincaré disk are all the Euclidean circle arcs (including diameters) that intersect the boundary circle $\partial\mathbb{D} = \{z \in \mathbb{C} \mid |z| = 1\}$ orthogonally.

Isometries. The group of orientation-preserving **isometries** of the Poincaré disk is:

$$\mathrm{PSU}(1,1) = \mathrm{PU}(1,1)$$

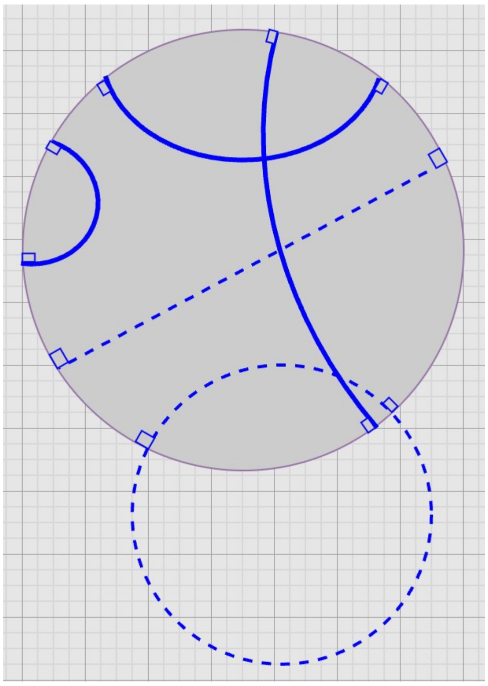

FIGURE 12. Geodesics on the Poincaré Disk Model

acting by Möbius (fractional linear) transformations:

$$z \mapsto \frac{az+b}{\bar{b}z+\bar{a}}, \quad \text{with } |a|^2 - |b|^2 = 1$$

Alternatively, an isometry can be written as:

$$z \mapsto u \cdot \frac{z-a}{1-\bar{a}z}, \quad \text{with } |u| = 1,\ |a| < 1$$

Orientation-reversing isometries replace z with $\bar{z}$.

Metric invariance. For an orientation-preserving disk isometry of the form

$$f(z) = e^{i\theta}\frac{z-a}{1-\bar{a}z}, \qquad |a| < 1,$$

we compute

$$|f'(z)|^2 = \frac{(1-|a|^2)^2}{|1-\bar{a}z|^4}$$

and

$$1 - |f(z)|^2 = \frac{(1-|a|^2)(1-|z|^2)}{|1-\bar{a}z|^2}.$$

Therefore,

$$\frac{4|f'(z)|^2|dz|^2}{(1-|f(z)|^2)^2} = \frac{4|dz|^2}{(1-|z|^2)^2}.$$

Hence

$$f^*(ds_D^2) = ds_D^2,$$

so these transformations preserve the Poincaré disk metric.

4.4. **Poincaré Half-Plane Model.** The Poincaré half-plane model is one of the most commonly used models of hyperbolic geometry. There are infinitely many parallel lines to a given line l through a point P not on l.

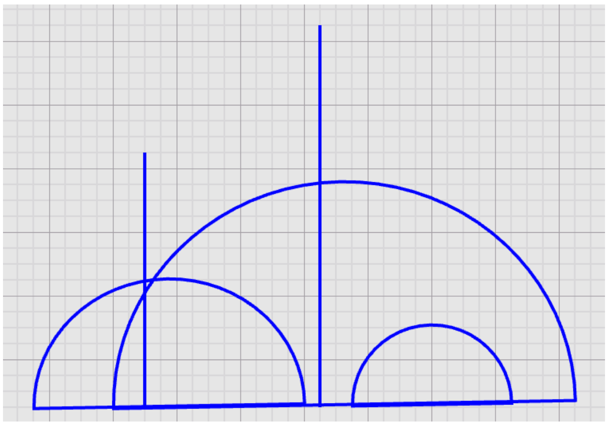

FIGURE 13. Upper Half-Plane Model

Definition 4.3. *Poincaré's upper half-plane denoted by $\mathbb{H}^2$ represents the upper half of the complex plane. It is described as:*

$$\mathbb{H}^2 = \{(x,y) \in \mathbb{R}^2 \mid y > 0\} = \{z \in \mathbb{C} \mid \operatorname{Im}(z) > 0\}$$

The topology of $\mathbb{H}^2$ is induced from $\mathbb{R}^2$ where $\mathbb{R}^2$ is endowed with usual topology. Thus, $\mathbb{H}^2$ inherits the subspace topology from $\mathbb{R}^2$ or, it is a subspace in $\mathbb{R}^2$.

Metric. *To define, Poincaré metric, a metric on this upper half-plane, $\mathbb{H}^2$, the line element, ds^2 is denoted as:*

$$ds^2 = \frac{dx^2 + dy^2}{y^2} = \frac{|dz|^2}{(\operatorname{Im}(z))^2}$$

It becomes singular as one approaches the real axis.

Christoffel symbols and geodesic equations. The upper half-plane metric is

$$ds^2 = \frac{dx^2 + dy^2}{y^2}.$$

Thus the metric tensor is

$$g_{ij} = \frac{1}{y^2}\delta_{ij}, \qquad g^{ij} = y^2\delta^{ij}.$$

The only nonzero derivatives of the metric components are

$$\frac{\partial g_{xx}}{\partial y} = \frac{\partial g_{yy}}{\partial y} = -\frac{2}{y^3}.$$

Using

$$\Gamma^k_{ij} = \frac{1}{2}g^{k\ell}\left(\frac{\partial g_{j\ell}}{\partial x_i} + \frac{\partial g_{i\ell}}{\partial x_j} - \frac{\partial g_{ij}}{\partial x_\ell}\right),$$

the nonzero Christoffel symbols are

$$\Gamma^x_{xy} = \Gamma^x_{yx} = -\frac{1}{y}, \qquad \Gamma^y_{xx} = \frac{1}{y}, \qquad \Gamma^y_{yy} = -\frac{1}{y}.$$

Therefore, for a curve $\gamma(t) = (x(t), y(t))$, the geodesic equations are

$$\ddot{x} - \frac{2\dot{x}\dot{y}}{y} = 0,$$

and

$$\ddot{y} + \frac{\dot{x}^2 - \dot{y}^2}{y} = 0.$$

Vertical lines satisfy these equations directly. If $x(t) = c$, then $\dot{x} = \ddot{x} = 0$, and the first equation is automatically satisfied. The second equation becomes

$$\ddot{y} - \frac{\dot{y}^2}{y} = 0,$$

which is solved by $y(t) = Ae^{Bt}$, the constant-speed parametrization of a vertical geodesic.

Semicircles orthogonal to the real axis also satisfy the equations. A semicircle with center $a \in \mathbb{R}$ and radius R may be parametrized by

$$x(t) = a + R \tanh t, \qquad y(t) = R \operatorname{sech} t.$$

Then

$$\dot{x} = R \operatorname{sech}^2 t, \qquad \dot{y} = -R \operatorname{sech} t \tanh t.$$

Substitution into the two geodesic equations gives

$$\ddot{x} - \frac{2\dot{x}\dot{y}}{y} = 0$$

and

$$\ddot{y} + \frac{\dot{x}^2 - \dot{y}^2}{y} = 0.$$

Thus vertical lines and semicircles orthogonal to the real axis are geodesics of the upper half-plane metric.

Suppose, we have a piecewise differentiable path $\alpha(t)$ in $\mathbb{H}^2$ expressed as

$$\alpha(t) = (x(t), y(t))$$

where $x(t)$ and $y(t)$ are functions of t, and $\alpha(t)$ lies in $\mathbb{H}^2$ so

$$\alpha(t) = (x(t), y(t)) \in \mathbb{H}^2 \subset \mathbb{R}^2$$

Since α is piecewise differentiable, both $x(t)$ and $y(t)$ are also piecewise differentiable. So, length of α, $L(\alpha)$ is

$$L(\alpha) = \int_a^b \frac{\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}}{y(t)} dt$$

Since $x(t)$ and $y(t)$ are piecewise differentiable , $\frac{dx}{dt}$ and $\frac{dy}{dt}$ exist, making the integral defined, and giving formal definition of the length of path $\alpha(t)$ in hyperbolic geometry. Consider path $\alpha(t) = (0, t)$ where t ranges from a to b so $\frac{dx}{dt} = 0$ and $\frac{dy}{dt} = 1$.

$$L(\alpha) = \int_a^b \frac{dt}{t} = \ln\left(\frac{b}{a}\right)$$

Thus, for a particular length α from $(0, a)$ to $(0, b)$ is given by $\ln(b/a)$. This length is referred to as the hyperbolic length of α. Complex notation:

$$\mathbb{H}^2 = \{z \in \mathbb{C} : \operatorname{Im}(z) > 0\}$$

$$ds^2 = \frac{|dz|^2}{(\operatorname{Im}(z))^2}$$

This is another way to express Poincaré metric. Here, dz represents the differential of the complex variable z, and it can be written as $dz = dx + idy$, where $z = x + iy$.

Distance Formula. The distance between two points in the model is given by:

$$d(z_1, z_2) = \operatorname{arcosh}\left(1 + \frac{|z_1 - z_2|^2}{2y_1y_2}\right).$$

This formula depends on the vertical positions y_1 and y_2 of the points.

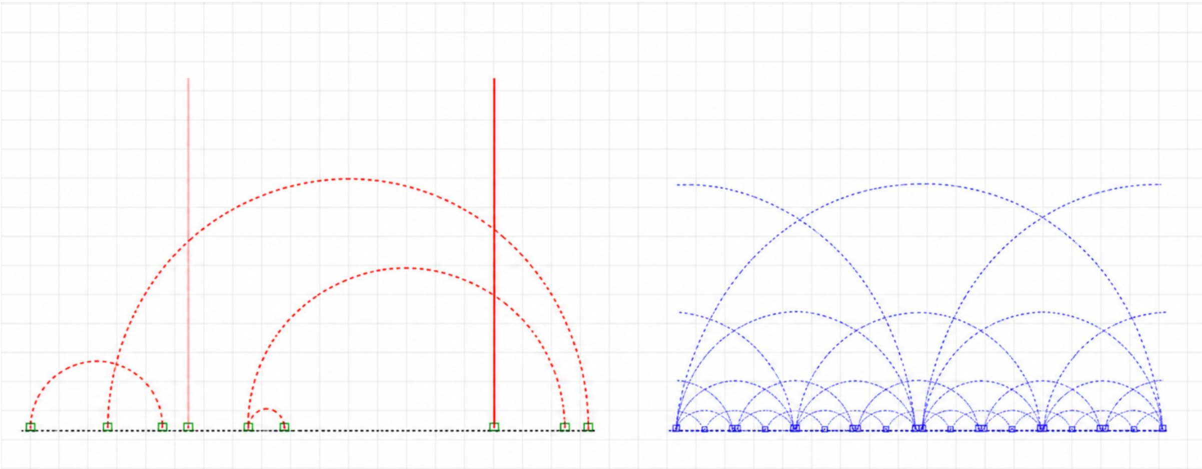

FIGURE 14. Geodesics on the Poincaré Half-Plane Model

Geodesics. Geodesics (shortest paths) are: Circular arcs orthogonal to the real axis, including vertical straight lines.

In the upper half-plane model, geodesics are precisely the vertical Euclidean lines and the Euclidean semicircles whose centers lie on the real axis. Equivalently, they are the lines and circles that meet the boundary $\mathbb{R} \cup \{\infty\}$ at right angles. Along a vertical geodesic from $x + iy_1$ to $x + iy_2$, the hyperbolic length is

$$\left|\ln \frac{y_2}{y_1}\right|.$$

The remaining geodesics are obtained by applying Möbius transformations that preserve the upper half-plane. Since Möbius transformations send generalized circles to generalized circles, the images of vertical lines are either vertical lines or semicircles orthogonal to the real axis. This explains why the geodesics in the upper half-plane have exactly this form.

Proposition 4.1. *In the upper half-plane model, the geodesics are precisely the vertical Euclidean lines and the Euclidean semicircles orthogonal to the real axis.*

Proof sketch. Vertical lines are geodesics because the length of a vertical path from $x+iy_1$ to $x + iy_2$ is

$$\int_{y_1}^{y_2} \frac{dy}{y} = \left|\ln \frac{y_2}{y_1}\right|.$$

Möbius transformations preserving the upper half-plane are hyperbolic isometries, and they send generalized circles to generalized circles. Therefore, the images of vertical geodesics under these isometries are either vertical lines or semicircles orthogonal to the real axis. Hence these curves give the geodesics of the upper half-plane. □

Isometries. The group of distance-preserving transformations is:

$$z \mapsto \frac{az + b}{cz + d}, \quad \text{with } a, b, c, d \in \mathbb{R},\ ad - bc = 1$$

These are fractional linear transformations from $\mathrm{PSL}(2, \mathbb{R})$. Orientation-reversing maps are obtained by conjugation.

4.5. **Comparison of the Classical models.** The following figure shows the relation between models in a hyperbolic space and the Geodesics in Poincaré, Klein and Hemisphere models.

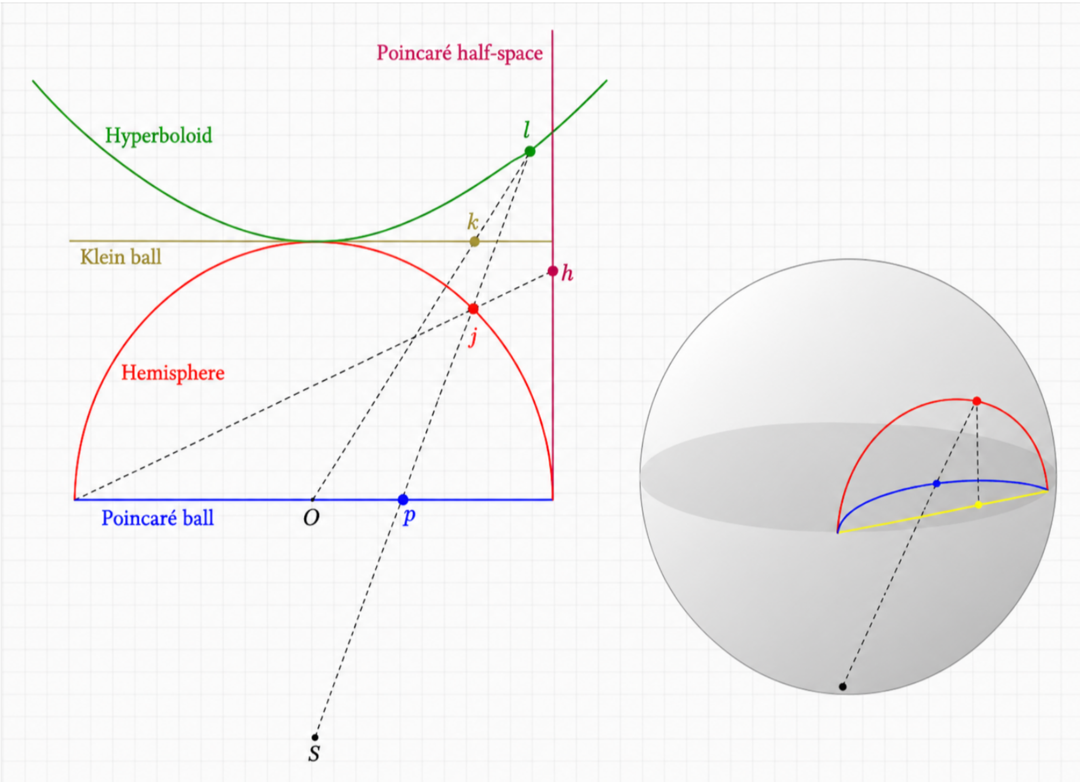

FIGURE 15. (a) Relationships among models of hyperbolic space. (b) Geodesics in the Poincaré ball, Klein ball, and hemisphere models. Reproduced from [4].

Why Multiple Models Represent the Same Geometry. Although the hyperboloid model, Poincaré disk model, Klein disk model, and upper half-plane model look different in Euclidean space, they represent the same hyperbolic plane. The difference lies in how the geometry is displayed, not in the underlying geometry itself.

The hyperboloid model is useful because it places hyperbolic geometry inside a linear algebraic setting using the Lorentzian metric. The Poincaré disk model is useful because it preserves angles and displays the entire hyperbolic plane inside a bounded circle. The Klein model is useful because geodesics appear as straight Euclidean chords, even though angles are distorted. The upper half-plane model is especially useful for studying Möbius transformations and the group of hyperbolic isometries.

Thus, each model emphasizes a different feature of the same object: the hyperboloid model emphasizes algebra, the Poincaré disk emphasizes conformal geometry, the Klein model emphasizes straight-line geodesics, and the upper half-plane emphasizes symmetry.

5. Hyperbolic Distance and Isometries

5.1. **Hyperbolic Distance:** Let z and w, be two points belonging to the upper half-plane $\mathbb{H}^2$. The **hyperbolic distance** between these two points, denoted by $d_{\mathbb{H}^2}(z, w)$, is defined as the infimum (the greatest lower bound) of the lengths of all piecewise differentiable paths γ connecting z and w. This is expressed as:

$$d_{\mathbb{H}^2}(z,w) = \inf\left\{L(\gamma) \mid \gamma \text{ is a piecewise differentiable path from } z \text{ to } w\right\}.$$

Proposition 5.1. *For $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$ in the upper half-plane model, the hyperbolic distance is*

$$d_{\mathbb{H}}(z_1, z_2) = \operatorname{arcosh}\left(1 + \frac{|z_1 - z_2|^2}{2y_1y_2}\right).$$

Proof sketch. The hyperbolic metric on the upper half-plane is

$$ds^2 = \frac{dx^2 + dy^2}{y^2}.$$

Along a vertical geodesic from $x + iy_1$ to $x + iy_2$, we have $dx = 0$, so the length is

$$L = \int_{y_1}^{y_2} \frac{dy}{y} = \left|\ln \frac{y_2}{y_1}\right|.$$

Möbius transformations in $PSL(2,\mathbb{R})$ preserve the hyperbolic metric and send geodesics to geodesics. Therefore, any geodesic semicircle orthogonal to the real axis can be mapped to a vertical geodesic. By applying such an isometry and rewriting the resulting vertical-distance expression in the original coordinates, one obtains

$$\cosh d_{\mathbb{H}}(z_1, z_2) = 1 + \frac{|z_1 - z_2|^2}{2y_1y_2}.$$

Hence,

$$d_{\mathbb{H}}(z_1, z_2) = \operatorname{arcosh}\left(1 + \frac{|z_1 - z_2|^2}{2y_1y_2}\right).$$

□

To tailor the distance in metric spaces, $d_{\mathbb{H}^2}$ to the hyperbolic context, we need to verify the three defining properties of a metric:

(1) **Non-negativity and identity of indiscernibles**: The distance between two points z and w in the upper half-plane is non-negative, and $d_{\mathbb{H}^2}(z,w) = 0$ if and only if $z = w$. This ensures that distinct points always have a positive distance between them.
(2) **Symmetry**: The distance function satisfies symmetry, meaning the distance from z to w is the same as the distance from w to z, or formally:
$$d_{\mathbb{H}^2}(z,w) = d_{\mathbb{H}^2}(w,z).$$
(3) **Triangle inequality**: The hyperbolic distance satisfies the triangle inequality, meaning that for any three points z, w, u in $\mathbb{H}^2$, we have:
$$d_{\mathbb{H}^2}(z,w) \leq d_{\mathbb{H}^2}(z,u) + d_{\mathbb{H}^2}(u,w).$$
This ensures that the direct distance between two points is always less than or equal to the sum of distances through any intermediate point. These properties together, prove that $d_{\mathbb{H}^2}$ is a valid metric, turning the upper half-plane $\mathbb{H}^2$ into a metric space.

This geometric description also illustrates the failure of Euclid's parallel postulate, already discussed earlier: through a point not on a given geodesic, there are infinitely many non-intersecting geodesics. Consider the upper half-plane model of hyperbolic geometry. Assume two points, z and w, that lie on a vertical line in this upper half-plane. Join z and w by a vertical line segment and call this path α. Compare it to any other path γ, where γ is also assumed to be piecewise differentiable, then the length of α is always less than or equal to the length of γ. In other words, α, the vertical line segment, is the shortest path between z and w. This makes α the geodesic between these two points.

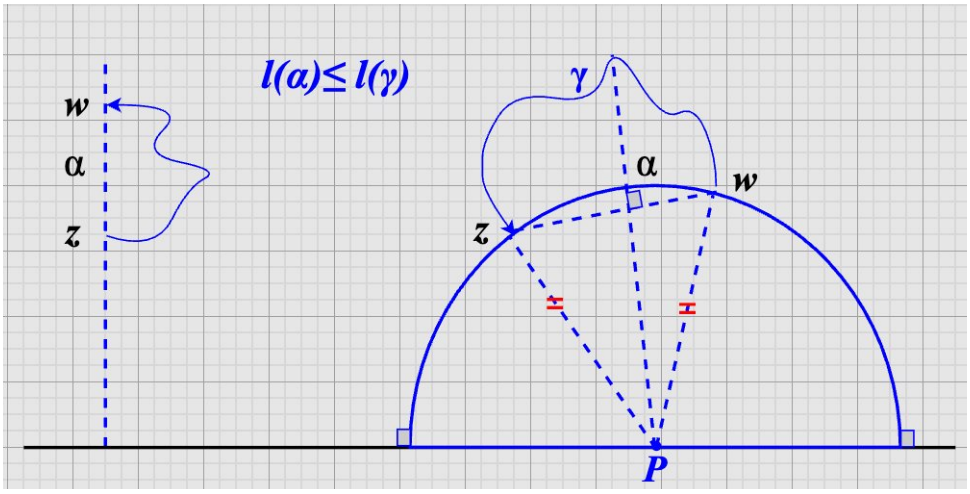

FIGURE 16. Shortest paths in hyperbolic geometry: vertical lines or portions of semicircles with center on Real axis

Next, consider the case where z and w do not lie on a vertical line. Join z and w by a Euclidean line segment and then draw the perpendicular bisector of this line that will meet the real axis (the x-axis) at some point, which we call p. The distances from p and z and from p and w will be equal in terms of Euclidean length. Using p as the center, we can now draw a semicircle that passes through both z and w, with a radius equal to the Euclidean distance from p and z.

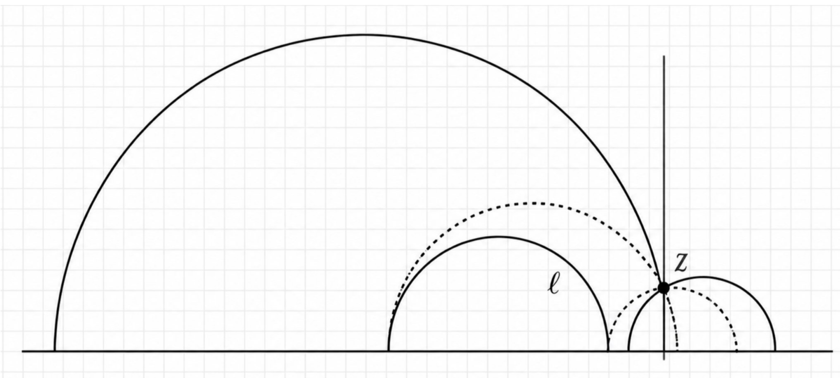

FIGURE 17. Violation of Fifth Axiom of Euclidean Geometry

In hyperbolic geometry, the geodesics are either vertical lines or arcs of semicircles centered on the real axis. These geodesics are the paths of the shortest distance between two points in the upper half-plane. Now, let us explore a fascinating property of hyperbolic geometry: if we take any point z in the upper half-plane and consider any geodesic that does not pass through z, it turns out there are infinitely many other geodesics that pass through z but do not intersect the original geodesic.
Geodesics are the paths of shortest distance between two points in the upper half-plane. They are either vertical lines or arcs of semicircles centered on the real axis. Take any point z in the upper half-plane and consider any geodesic that does not pass through z , it turns out there are infinitely many other geodesics that pass through z but do not intersect the original geodesic, l. If l is a vertical line, infinitely many semicircles centered

on the real axis pass through z without intersecting l In hyperbolic geometry, however, there are infinitely many such lines, or in this case, geodesics, that do not intersect the given geodesic l.

5.2. **Isometry:**

Definition 5.1. *Let $T : \mathbb{H}^2 \to \mathbb{H}^2$ be a map expressed as $T(x, y) = u(x, y), v(x, y)$. For T to be an **isometry**, it must preserve hyperbolic metric and the following must hold:*

$$\frac{dx^2 + dy^2}{y^2} = \frac{du^2 + dv^2}{v^2}$$

Example 1. *Consider the map $T : \mathbb{H}^2 \to \mathbb{H}^2$ defined by*

$$T(x, y) = (x + a, y)$$

where a is a real number. In this case, the map shifts x-coordinate by a constant a, while leaving y-coordinate unchanged. So, we have

$$u(x, y) = x + a \quad \textit{and} \quad v(x, y) = y$$

For this transformation, the differentials $du = dx$ and $dv = dy$, so

$$\frac{du^2 + dv^2}{v^2} = \frac{dx^2 + dy^2}{y^2}$$

Hence, this transformation T preserves the hyperbolic metric, and we conclude that T is an isometry.

Example 2. *Consider the transformation $T(x, y)$ defined by*

$$T(x, y) = (\lambda x, \lambda y)$$

where λ is a positive real number. Under this transformation, the component functions become

$$u(x, y) = \lambda x \ \textit{and} \quad v(x, y) = \lambda y$$

Differentiating, we get:

$$du = \lambda dx \qquad \textit{and } dv = \lambda dy$$

Substituting into the hyperbolic metric expression, we get:

$$\frac{du^2 + dv^2}{v^2} = \frac{\lambda^2(dx^2 + dy^2)}{\lambda^2 y^2} = \frac{dx^2 + dy^2}{y^2}$$

*Thus, T preserves the hyperbolic distance and the transformation T is an **isometry** of the upper-half plane.*

The map $T(x, y) = (x + a, y)$ performs horizontal translation and shifts each point along x-axis and the map $T(x, y) = (\lambda x, \lambda y)$ scales both coordinates by a positive scalar λ, radially away from or towards the origin. Both transformations leave the hyperbolic line element unchanged and are thus examples of **isometries** in the upper half-plane model.

Action of $\mathrm{SL}(2, \mathbb{R})$ **on** $\mathbb{H}^2$ **:** Examine the action of $\mathrm{SL}(2, \mathbb{R})$ on the upper half-plane, $\mathbb{H}^2$. The $\mathrm{SL}(2, \mathbb{R})$ consists of all 2×2 real matrices of the form

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad \text{with} \quad ad - bc = 1,$$

where $a, b, c, d \in \mathbb{R}$.

$$\mathrm{SL}(2, \mathbb{R}) \times \mathbb{H}^2 \to \mathbb{H}^2$$
$$\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, z \right) \to \frac{az + b}{cz + d}$$

If $z \in \mathbb{H}^2$, then $T(z) = \frac{az+b}{cz+d} \in \mathbb{H}^2$, where $a, b, c, d \in \mathbb{R}$ and $ad - bc = 1$. The imaginary part of $T(z)$ can be expressed as:

$$\operatorname{Im}(T(z)) = \frac{\operatorname{Im}(z)}{|cz+d|^2} > 0,$$

which confirms that $T(z) \in \mathbb{H}^2$. The group $\mathrm{SL}(2, \mathbb{R})$ acts on $\mathbb{H}^2$ through Möbius transformations. So, T is a *homeomorphism* and establishes a continuous one-to-one mapping from $\mathbb{H}^2$ to itself.

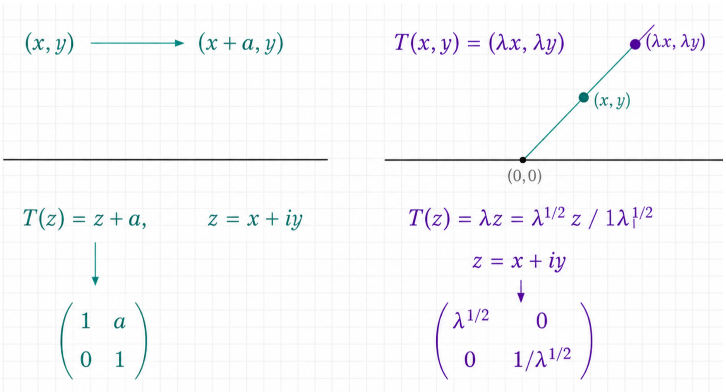

Figure 18. Transformation

Action of $\mathrm{PSL}(2, \mathbb{R})$ **on** $\mathbb{H}^2$ **:**

Definition 5.2. *Let* $\mathbb{H}^2 = \{z \in \mathbb{C} \mid Im(z) > 0\}$ *denote the upper half plane. The group* $\mathrm{PSL}(2, \mathbb{R})$ *is defined as the quotient group:*

$$\mathrm{PSL}(2, \mathbb{R}) := \mathrm{SL}(2, \mathbb{R})/\{\pm I\}$$

where $\mathrm{SL}(2, \mathbb{R})$ *is the group of all* 2×2 *real matrices with determinant 1, and* I *is the identity matrix. We define a map*

$$\varphi : \mathrm{PSL}(2, \mathbb{R}) \to Isom^+(\mathbb{H}^2)$$

by letting φ *act on* $z \in \mathbb{H}^2$ *as a Möbius transformation:*

$$\varphi\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right)(z) = \frac{az+b}{cz+d}.$$

Theorem 5.1. *The map* φ *defines an injective group homomorphism from* $\mathrm{PSL}(2, \mathbb{R})$ *into the group of orientation-preserving isometries of* $\mathbb{H}^2$*.*

Proof. Let $z = x + iy \in \mathbb{H}^2$, with $y > 0$. Consider the Möbius transformation

$$T(z) = \frac{az+b}{cz+d},$$

where $a, b, c, d \in \mathbb{R}$ and $ad - bc = 1$.

We first verify that $T(z) \in \mathbb{H}^2$. The imaginary part of $T(z)$ is

$$\operatorname{Im}\left(\frac{az+b}{cz+d}\right) = \frac{\operatorname{Im}(z)}{|cz+d|^2}.$$

Since $\operatorname{Im}(z) > 0$ and $|cz+d|^2 > 0$, it follows that

$$\operatorname{Im}(T(z)) > 0,$$

so the image remains in $\mathbb{H}^2$.

Now, we show that T preserves the hyperbolic metric. Let

$$\gamma(t) = x(t) + iy(t), \qquad t \in [\alpha, \beta],$$

be a piecewise differentiable path in $\mathbb{H}^2$. The hyperbolic length of γ is

$$L(\gamma) = \int_\alpha^\beta \frac{\sqrt{x'(t)^2 + y'(t)^2}}{y(t)}\, dt.$$

The complex derivative of T is

$$T'(z) = \frac{ad - bc}{(cz+d)^2} = \frac{1}{(cz+d)^2}.$$

The imaginary part transforms as

$$\mathrm{Im}(T(z)) = \frac{\mathrm{Im}(z)}{|cz+d|^2} = \frac{y}{|cz+d|^2}.$$

Combining these two identities, we get

$$\frac{|dT(z)|^2}{\mathrm{Im}(T(z))^2} = \frac{|T'(z)|^2|dz|^2}{\mathrm{Im}(T(z))^2} = \frac{|dz|^2/|cz+d|^4}{y^2/|cz+d|^4} = \frac{|dz|^2}{y^2}.$$

Thus the hyperbolic metric

$$ds^2 = \frac{|dz|^2}{(\mathrm{Im}\, z)^2}$$

is unchanged under T.

Also,

$$|T'(z)| = \frac{1}{|cz+d|^2} = \frac{\mathrm{Im}(T(z))}{\mathrm{Im}(z)}.$$

Therefore, for $z = \gamma(t)$, we find

$$|T'(\gamma(t))| \cdot |\gamma'(t)| = \frac{\mathrm{Im}(T(\gamma(t)))}{\mathrm{Im}(\gamma(t))} \cdot |\gamma'(t)|.$$

Thus, the length of the image path is

$$L(T \circ \gamma) = \int_\alpha^\beta \frac{|T'(\gamma(t))| \cdot |\gamma'(t)|}{\mathrm{Im}(T(\gamma(t)))}\, dt = \int_\alpha^\beta \frac{|\gamma'(t)|}{\mathrm{Im}(\gamma(t))}\, dt = L(\gamma).$$

Hence, T preserves hyperbolic length and is an isometry of $\mathbb{H}^2$.

Because Möbius transformations with real coefficients and determinant one are orientation-preserving transformations of $\mathbb{H}^2$, and matrix multiplication corresponds to composition of Möbius maps, the map is a group homomorphism.

Finally, we check injectivity. Let

$$\psi : \mathrm{SL}(2, \mathbb{R}) \to \mathrm{Isom}^+(\mathbb{H}^2)$$

be given by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \left(z \mapsto \frac{az+b}{cz+d} \right).$$

Suppose $\psi(A)$ is the identity map. Then

$$\frac{az+b}{cz+d} = z$$

for all $z \in \mathbb{H}^2$. Hence,

$$az + b = cz^2 + dz,$$

so

$$cz^2 + (d-a)z - b = 0 \qquad \forall z \in \mathbb{H}^2.$$

This implies that the coefficients must vanish. Thus,

$$c = 0, \qquad b = 0, \qquad d = a.$$

Also, $ad - bc = 1$ implies $a^2 = 1$, so $a = \pm 1$. Therefore,

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{or} \quad A = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Hence,

$$\ker \psi = \{\pm I\}.$$

Since the matrices A and $-A$ define the same fractional linear transformation, the action factors through

$$\mathrm{PSL}(2, \mathbb{R}) = \mathrm{SL}(2, \mathbb{R})/\{\pm I\}.$$

Therefore, φ is injective, and $\mathrm{PSL}(2, \mathbb{R})$ embeds as a subgroup of $\mathrm{Isom}^+(\mathbb{H}^2)$. □

Fuchsian Group. A discrete subgroup of $PSL(2, \mathbb{R})$ is called a Fuchsian group. Such groups act on the upper half-plane by hyperbolic isometries, and their orbits can be used to generate repeating geometric patterns. If a suitable polygon $F \subset \mathbb{H}$ is chosen as a fundamental domain, then the images of F under the group action can tile the hyperbolic plane. This idea connects the isometry group of the upper half-plane to the regular tessellations discussed later.

5.3. **Möbius Transformations and the Isometry Group.** The isometries of the upper half-plane model can be described algebraically using Möbius transformations. A real Möbius transformation has the form

$$T(z) = \frac{az + b}{cz + d}, \qquad a, b, c, d \in \mathbb{R}, \qquad ad - bc = 1.$$

Such transformations are represented by matrices in $SL(2, \mathbb{R})$:

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

For $z \in \mathbb{H}$, one computes

$$\mathrm{Im}(T(z)) = \frac{\mathrm{Im}(z)}{|cz + d|^2}.$$

Since $\mathrm{Im}(z) > 0$, this shows that T maps the upper half-plane to itself. Also,

$$T'(z) = \frac{1}{(cz + d)^2}.$$

Therefore,

$$\frac{|T'(z)|^2 |dz|^2}{\mathrm{Im}(T(z))^2} = \frac{|dz|^2/|cz + d|^4}{\mathrm{Im}(z)^2/|cz + d|^4} = \frac{|dz|^2}{\mathrm{Im}(z)^2}.$$

Hence T preserves the hyperbolic metric

$$ds^2 = \frac{|dz|^2}{(\mathrm{Im}\, z)^2}.$$

Thus real Möbius transformations with determinant one are orientation-preserving isometries of the upper half-plane.

The matrices A and $-A$ define the same Möbius transformation, so the orientation-preserving isometry group of the upper half-plane is

$$PSL(2, \mathbb{R}) = SL(2, \mathbb{R})/\{\pm I\}.$$

This explains why Möbius transformations are central to hyperbolic geometry: they provide the algebraic form of the symmetries of the hyperbolic plane.

6. Applications of Hyperbolic Geometry

Hyperbolic geometry has diverse applications in the field of mathematics, and multiple modern scientific domains. In Special Relativity, the spacetime geometry can be interpreted using hyperbolic models. Hyperbolic geometry has also influenced art, design, and architecture through its visual language of curvature, symmetry, and tessellation. The Poincaré disk model provides a natural framework for hyperbolic tessellations, as seen in M.C. Escher's circle-limit works. In architecture, forms with negative curvature and saddle-like surfaces sometimes appear in visually striking structures, although these are usually inspirations from hyperbolic geometry rather than direct models of the hyperbolic plane.

Hyperbolic geometry has become a powerful tool in computer science. It is used to analyze and visualize complex networks and data structures. It has enhanced machine learning tasks, thereby providing high-dimensional data while preserving proximity relationships and is valuable in graph-based algorithms, clustering etc., where Euclid methods fail. Hyperbolic embeddings are used in machine learning and network analysis because they can represent hierarchical and tree-like data efficiently.

In theoretical physics, it serves as a foundational network for studying curvature, symmetry, and geometric transformations. Hyperbolic geometry is also used to investigate minimal surfaces, curvature flows, and variational problems. By integrating hyperbolic geometry into curricula, educators foster deeper spatial reasoning, preparing students to tackle modern challenges in data science, physics and beyond.

6.1. Angle Defect and Regular Hyperbolic Tessellations.

Theorem 6.1 (Gauss–Bonnet for hyperbolic polygons)**.** *Let P be a geodesic n-gon in the hyperbolic plane with interior angles $\alpha_1, \alpha_2, \ldots, \alpha_n$. Then*

$$\operatorname{Area}(P) = (n-2)\pi - \sum_{i=1}^{n} \alpha_i.$$

Proof sketch. Since the sides of P are geodesics, their geodesic curvature is zero. The hyperbolic plane has constant Gaussian curvature $K = -1$. Applying the Gauss–Bonnet theorem to the polygon gives

$$-\operatorname{Area}(P) + \sum_{i=1}^{n} (\pi - \alpha_i) = 2\pi.$$

Rearranging,

$$\operatorname{Area}(P) = (n-2)\pi - \sum_{i=1}^{n} \alpha_i.$$

□

One of the most striking consequences of negative curvature is that the hyperbolic plane admits infinitely many regular tessellations. A regular $\{p, q\}$ tessellation consists of regular p-gons, with exactly q polygons meeting at every vertex.

Theorem 6.2. *For a regular hyperbolic $\{p, q\}$ tessellation, the parameters satisfy*

$$\frac{1}{p} + \frac{1}{q} < \frac{1}{2}.$$

Proof. In a regular $\{p, q\}$ tessellation, each tile is a regular p-gon and q polygons meet at each vertex. Therefore, the interior angle of each polygon is

$$\alpha = \frac{2\pi}{q}.$$

Using the hyperbolic polygon area formula,

$$\text{Area} = (p-2)\pi - p\alpha = (p-2)\pi - p \cdot \frac{2\pi}{q}.$$

Since area must be positive in the hyperbolic plane,

$$(p-2)\pi - \frac{2p\pi}{q} > 0.$$

Dividing by π, we get

$$p - 2 - \frac{2p}{q} > 0.$$

Rearranging,

$$\frac{1}{p} + \frac{1}{q} < \frac{1}{2}.$$

□

The converse is also true: whenever

$$\frac{1}{p} + \frac{1}{q} < \frac{1}{2},$$

there exists a regular hyperbolic $\{p, q\}$ tessellation. The idea is to start with a hyperbolic triangle whose angles are

$$\frac{\pi}{p}, \qquad \frac{\pi}{q}, \qquad \frac{\pi}{2}.$$

Such a triangle exists because

$$\frac{\pi}{p} + \frac{\pi}{q} + \frac{\pi}{2} < \pi,$$

which is equivalent to

$$\frac{1}{p} + \frac{1}{q} < \frac{1}{2}.$$

This is impossible in Euclidean geometry, where the angles of a triangle must add to exactly π, but it is possible in the hyperbolic plane because triangle angle sums are strictly less than π.

Reflecting this triangle across its sides produces congruent copies of the triangle. Repeated reflections generate a reflection group, and the reflected copies tile the hyperbolic plane without gaps or overlaps. By assembling $2p$ such triangles around the vertex with angle $\frac{\pi}{p}$, one obtains a regular p-gon. The angle $\frac{\pi}{q}$ appears twice at each vertex of this p-gon, so each interior angle is

$$\frac{2\pi}{q}.$$

Therefore, exactly q such p-gons meet around each vertex, producing a regular hyperbolic $\{p, q\}$ tessellation. A complete proof of this construction uses reflection groups or the Poincaré polygon theorem.

This condition contrasts sharply with the Euclidean case. In Euclidean geometry, regular tessellations satisfy

$$\frac{1}{p} + \frac{1}{q} = \frac{1}{2},$$

which gives only three regular tilings:

$$\{3,6\}, \qquad \{4,4\}, \qquad \{6,3\}.$$

These correspond to equilateral triangles, squares, and regular hexagons. In hyperbolic geometry, the strict inequality

$$\frac{1}{p}+\frac{1}{q}<\frac{1}{2}$$

allows infinitely many possibilities, such as

$$\{5,4\}, \qquad \{7,3\}, \qquad \{8,3\}, \qquad \{5,5\}.$$

This abundance of regular tessellations reflects the extra space created by negative curvature. Thus, the geometry of the hyperbolic plane is not only different from Euclidean geometry at the level of parallel lines and triangle angle sums; it also supports a much richer world of regular patterns.

This contrasts with Euclidean geometry, where regular tessellations satisfy

$$\frac{1}{p}+\frac{1}{q}=\frac{1}{2}.$$

The Euclidean equality allows only $\{3,6\}$, $\{4,4\}$, and $\{6,3\}$, while the hyperbolic inequality allows infinitely many possibilities. This shows how negative curvature creates more room for regular patterns than flat geometry does.

7. Future Directions

The discussion in this paper focused on the hyperbolic plane through its curvature, models, geodesics, and isometries. Several mathematical directions naturally extend from this approach.

7.1. **Curvature of General Conformal Metrics.** In this paper, the curvature of the upper half-plane model was verified using the conformal metric

$$ds^2=\lambda^2(dx^2+dy^2), \qquad \lambda=\frac{1}{y}.$$

For such a metric, the Gaussian curvature is

$$K=-\frac{\Delta(\log\lambda)}{\lambda^2}.$$

A natural extension is to study other conformal factors $\lambda(x,y)$ and determine when they produce constant negative curvature. This leads to the equation

$$\Delta(\log\lambda)=\lambda^2,$$

which characterises conformal metrics of curvature (K=-1). Studying this equation would give a deeper analytic understanding of why the Poincar'e disk and upper half-plane models represent the same constant-curvature geometry.

7.2. **Geodesic Equations in Different Models.** Another direction is to derive the geodesic equations in each model directly from the metric. In the upper half-plane model, the Christoffel symbols lead to the equations

$$\ddot{x}-\frac{2\dot{x}\dot{y}}{y}=0, \qquad \ddot{y}+\frac{\dot{x}^2-\dot{y}^2}{y}=0.$$

A further project would be to repeat this computation for the Poincar'e disk metric

$$ds^2=\frac{4(dx^2+dy^2)}{(1-x^2-y^2)^2}$$

and compare the resulting differential equations with the known fact that disk geodesics are arcs orthogonal to the boundary circle. This would connect the visual description of geodesics with their differential-geometric definition.

7.3. **Metric Pullbacks Between Models.** The hyperboloid model, Poincar'e disk model, Beltrami–Klein model, and upper half-plane model are equivalent, but their equivalence is most meaningful when the metric is tracked explicitly. This paper verified that stereographic projection from the hyperboloid gives the Poincar'e disk metric. A further direction is to compute other pullbacks explicitly, such as the central projection from the hyperboloid to the Beltrami–Klein disk and the Cayley transform between the disk and the upper half-plane. This would show precisely how the same hyperbolic metric changes form under different coordinate systems.

7.4. **Higher-Dimensional Hyperbolic Space.** The hyperboloid construction extends naturally to higher dimensions. Hyperbolic n-space is defined by

$$\mathbb{H}^n = \left\{ x \in \mathbb{R}^{n,1} \;\middle|\; -x_0^2 + x_1^2 + \cdots + x_n^2 = -1, \quad x_0 > 0 \right\}.$$

The ambient Lorentzian metric is

$$ds^2 = -dx_0^2 + dx_1^2 + \cdots + dx_n^2.$$

Its restriction to the tangent spaces of $\mathbb{H}^n$ induces the hyperbolic metric on this space, and its isometry group is related to

$$O^+(n, 1).$$

This extension would connect the two-dimensional models studied in the paper to higher-dimensional geometry and to the Lorentzian structures used in relativity.

8. Conclusion

Hyperbolic geometry begins with a single change to Euclid's parallel postulate, but this change produces a geometry with its own distance, curvature, symmetries, and visual structure. Unlike the Euclidean plane, the hyperbolic plane has constant negative curvature, triangles with angle sum less than π, and distances that behave very differently near the boundaries of its standard models.

The different models of the hyperbolic plane are not separate geometries, but different representations of the same underlying space. The hyperboloid model emphasizes the role of curvature and the Lorentzian metric, stereographic projection connects the hyperboloid model to the Poincaré disk, and the upper half-plane model reveals the importance of Möbius transformations. Together, these models show that hyperbolic geometry can be understood through curvature, distance, projection, and group actions.

Möbius transformations demonstrate how symmetry organizes the hyperbolic plane. In the upper half-plane model, transformations of the form

$$z \mapsto \frac{az + b}{cz + d}$$

with $ad - bc = 1$ preserve the hyperbolic metric and act as orientation-preserving isometries. This connects hyperbolic geometry naturally to the group

$$PSL(2, \mathbb{R}).$$

Finally, the angle defect formula explains why hyperbolic geometry admits infinitely many regular tessellations. In Euclidean geometry, regular tilings are limited because angles must fit exactly in a flat plane. In hyperbolic geometry, negative curvature creates additional space, allowing the condition

$$\frac{1}{p} + \frac{1}{q} < \frac{1}{2}$$

to produce infinitely many regular tessellations $\{p, q\}$. Thus, hyperbolic geometry is not merely an alternative to Euclidean geometry, but a rich and internally consistent mathematical system with its own models, symmetries, and patterns.

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