Stokes' Theorem and de Rham Cohomology through the Language of Forms

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Abstract

The Fundamental Theorem of Calculus, Green's theorem, the classical Stokes theorem, and the divergence theorem each relate differentiation on a region to integration on its boundary. This paper develops the language of smooth manifolds and differential forms in which these results become instances of a single statement. After introducing tangent and cotangent spaces, alternating forms, the wedge product, and the exterior derivative, we define integration of top-degree forms on oriented manifolds. We then state the generalized Stokes theorem and give a proof sketch assembled globally by a partition of unity. Finally, we introduce closed and exact forms and de Rham cohomology. The standard angular form in the punctured plane gives a concrete example of a closed form that is not exact, showing how cohomology detects topological holes.

1 Introduction

Calculus contains a recurring pattern:

$$\int_{\text{region}} (\text{derivative of object}) = \int_{\text{boundary of the region}} (\text{original object}).$$

The Fundamental Theorem of Calculus says

$$\int_a^b f'(x)\,dx = f(b) - f(a).$$

The right-hand side can be regarded as an integral over the oriented boundary of the interval: the endpoint b has sign $+1$ and the endpoint a has sign -1. In higher-dimensional calculus the same pattern appears in the forms

$$\begin{aligned} \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA &= \oint_{\partial D} P\,dx + Q\,dy && \text{(Green's theorem)}, \\ \iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n}\,dS &= \oint_{\partial S} \mathbf{F} \cdot d\mathbf{r} && \text{(classical Stokes)}, \\ \iiint_V (\nabla \cdot \mathbf{F})\,dV &= \iint_{\partial V} \mathbf{F} \cdot \mathbf{n}\,dS && \text{(divergence theorem)}. \end{aligned}$$

Each theorem differentiates something on a region and integrates the result; the original object is then integrated on the boundary. The purpose of this paper is to explain the common structure behind these theorems. The discussion is aimed at readers who know multivariable calculus and are meeting manifolds and differential forms for the first time; the emphasis is therefore on geometric meaning and the calculations that recover the familiar theorems. The main result is the following.

Theorem 1.1 (Generalized Stokes theorem)**.** *Let M be a compact oriented smooth n-manifold with boundary, and let $\omega \in \Omega^{n-1}(M)$. Then*

$$\int_M d\omega = \int_{\partial M} \omega,$$

where ∂M has the boundary orientation.

The notation in Theorem 1.1 will be developed in the next sections. The route mirrors the familiar calculus story: manifolds supply domains for calculus, differential forms supply integrands, and the exterior derivative supplies a coordinate-independent derivative. In particular, the formal objects needed to unify the four displayed theorems are: spaces on which calculus can be performed, objects that can be integrated over those spaces, and a generalized differentiation operation.

2 Smooth manifolds

2.1 Topological background

Before introducing manifolds, we recall just the topological language needed for their definition. It makes precise what it means for a space to be locally Euclidean.

Definition 2.1. A *topology* on a set X is a collection $\mathcal{T}$ of subsets of X, called *open sets*, such that:

(i) $\varnothing$ and X belong to $\mathcal{T}$;

(ii) the union of any collection of sets in $\mathcal{T}$ belongs to $\mathcal{T}$;

(iii) the intersection of finitely many sets in $\mathcal{T}$ belongs to $\mathcal{T}$.

The pair $(X, \mathcal{T})$ is a *topological space*. An open set U containing p is called an *open neighborhood* of p.

In $\mathbb{R}^n$, the usual topology is generated by open balls. A manifold uses the same notion of local neighborhood, but it need not be a global subset of a single $\mathbb{R}^n$.

Definition 2.2. A topological space X is *Hausdorff* if, for every two distinct points $p, q \in X$, there are disjoint open sets U and V with $p \in U$ and $q \in V$.

The Hausdorff condition ensures that distinct points can be separated by neighborhoods, as they can in Euclidean space.

Definition 2.3. A collection $\mathcal{B}$ of open subsets of X is a *basis* for the topology if every open set is a union of members of $\mathcal{B}$. The space is *second countable* if it has a countable basis.

For example, open balls with rational centers and radii form a countable basis of $\mathbb{R}^n$. This mild condition holds for the usual geometric examples and supports the global constructions used later, such as partitions of unity.

Definition 2.4. A map $f : X \to Y$ between topological spaces is *continuous* if $f^{-1}(U)$ is open in X whenever U is open in Y. A bijection $f : X \to Y$ is a *homeomorphism* if both f and f^{-1} are continuous.

A homeomorphism may bend or stretch a space, but cannot tear it apart or glue points together. Thus local homeomorphism to $\mathbb{R}^n$ is the rigorous meaning of "locally Euclidean."

2.2 Charts, atlases, and smoothness

Definition 2.5. An *n-dimensional topological manifold* is a Hausdorff, second-countable topological space M such that every $p \in M$ has an open neighborhood U and a homeomorphism $\varphi : U \to V$, where $V \subseteq \mathbb{R}^n$ is open. The pair (U, φ) is a *chart*.

The set $U \subset M$ is called a *coordinate patch*; its image $V = \varphi(U) \subseteq \mathbb{R}^n$ is the *coordinate domain.* A chart is therefore a local translation between the manifold and a familiar Euclidean region. The coordinates $x^1, \ldots, x^n$ on V can be used to do calculus on U. One chart usually cannot describe the entire manifold: for example, the sphere has no single chart covering all of S^2, but every point of the sphere has a neighborhood that can be flattened into a planar coordinate domain.

Definition 2.6. An *atlas* for a manifold M is a collection of charts $\{(U_i, \varphi_i)\}_{i \in I}$ whose patches cover M. If two patches overlap, the map

$$\varphi_j \circ \varphi_i^{-1} : \varphi_i(U_i \cap U_j) \longrightarrow \varphi_j(U_i \cap U_j)$$

is the *transition map* from the ith coordinates to the jth coordinates.

A topological manifold is *smooth* when it has an atlas whose transition maps are C^∞ (infinitely differentiable). Smooth transition maps ensure that a derivative computed in one chart agrees with the derivative computed in another. Thus a manifold may be curved globally while looking like Euclidean space in every sufficiently small chart, with all local calculations fitting together consistently.

The familiar examples illustrate the local-versus-global distinction. A smooth curve is a one-dimensional manifold, and a smooth surface is a two-dimensional manifold. The sphere S^2 and a torus are both locally modelled on open subsets of $\mathbb{R}^2$, even though neither is globally a single copy of the plane. The charts and their transition maps provide the local coordinates needed to perform calculus without pretending that the entire space is flat.

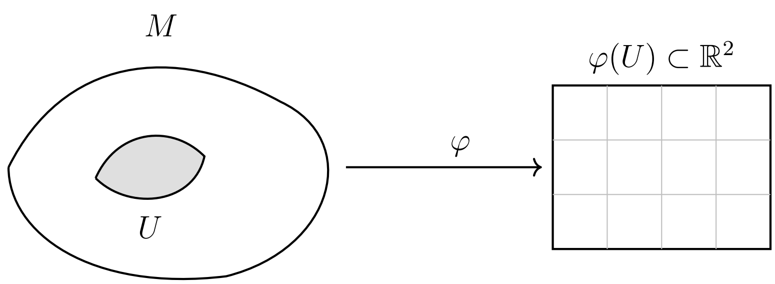


Figure 1: A chart identifies a local patch of a manifold with an open Euclidean set.

Manifolds with boundary are defined similarly, except that local charts are allowed to map into the half-space $\mathbb{H}^n = \{x \in \mathbb{R}^n : x_n \geq 0\}$. Points mapped to $x_n = 0$ form ∂M. This is the setting required by Stokes' theorem.

3 Tangent vectors and covectors

Let $p \in M$. A tangent vector at p can be represented by the velocity $\gamma'(0)$ of a smooth curve $\gamma : (-\varepsilon, \varepsilon) \to M$ with $\gamma(0) = p$, where two curves represent the same vector if their coordinate velocities agree in one, hence every, chart. The resulting vector space is the *tangent space* T_pM.

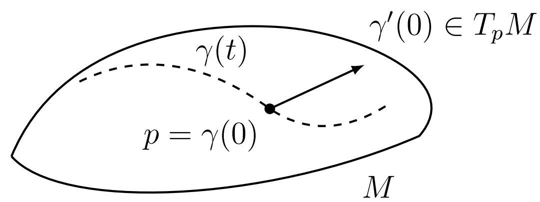


Figure 2: A tangent vector $v = \gamma'(0) \in T_pM$ is the velocity of a curve γ passing through p.

In coordinates $(x^1, \dots, x^n)$, it has basis

$$\left.\frac{\partial}{\partial x^1}\right|_p, \dots, \left.\frac{\partial}{\partial x^n}\right|_p.$$

Thus, in a two-dimensional coordinate patch, a tangent vector has the form

$$v = P\frac{\partial}{\partial x} + Q\frac{\partial}{\partial y}.$$

Definition 3.1. The *cotangent space* at p is the dual vector space

$$T_p^*M = \{\omega : T_pM \to \mathbb{R} : \omega \text{ is linear}\}.$$

Its elements are called *covectors.*

The dual basis is written $dx^1, \dots, dx^n$ and satisfies

$$dx^i\left(\frac{\partial}{\partial x^j}\right) = \delta^i_j.$$

For instance, in $\mathbb{R}^2$, if we have a vector $v = P\,\partial_x + Q\,\partial_y$ and covector $\omega = M\,dx + N\,dy$, then we see their pairing is a scalar.

$$\omega(v) = MP + NQ.$$

This distinction matters: vectors give directions, while covectors measure vectors and return numbers. In particular,

$$dx(\partial_x) = 1, \qquad dx(\partial_y) = 0, \qquad dy(\partial_x) = 0, \qquad dy(\partial_y) = 1.$$

These identities show that dx and dy extract the x- and y-components of a tangent vector. Consequently the coordinate expression $M\,dx + N\,dy$ is not merely suggestive notation: it is exactly the covector whose value on $P\partial_x + Q\partial_y$ is $MP + NQ$.

4 Differential forms and the exterior derivative

Definition 4.1. A multilinear map takes several vectors as inputs and is linear in each input separately. It is *alternating* if it changes sign when two inputs are swapped; in particular, it is zero whenever two inputs are equal. A *k-form* at p is an alternating k-linear map $(T_pM)^k \to \mathbb{R}$. A smooth assignment of a k-form to each $p \in M$ is a *differential k-form*; the vector space of such forms is denoted $\Omega^k(M)$.

The wedge product combines an r-form α and an s-form β into an $(r+s)$-form. It is bilinear and graded-commutative:

$$\alpha \wedge \beta = (-1)^{rs} \beta \wedge \alpha.$$

In particular, $dx \wedge dy = -dy \wedge dx$ and $\alpha \wedge \alpha = 0$ for any form α. The form $dx \wedge dy$ represents an oriented area element; $dx \wedge dy \wedge dz$ represents an oriented volume element.

Definition 4.2. The degree of a differential form α, denoted $\deg(\alpha)$, is the number k such that $\alpha \in \Omega^k(M)$.

Example 4.3 (Forms of different degrees)**.** On a coordinate region in $\mathbb{R}^3$, the following are respectively a 0-, 1-, 2-, and 3-form:

$$x^2 \sin(yz), \qquad 2\,dx + x\,dy + z^2\,dz, \qquad \sin(z)\,dx \wedge dy, \qquad xyz\,dx \wedge dy \wedge dz.$$

A k-form is the type of object that can be integrated over an oriented k-dimensional domain. This degree matching is the differential-form version of choosing dx, dA, or dV in elementary calculus.

Every k-form on a coordinate neighborhood has the form

$$\omega = \sum_{i_1 < \cdots < i_k} f_{i_1 \cdots i_k}\, dx^{i_1} \wedge \cdots \wedge dx^{i_k}.$$

The *exterior derivative* is the linear map

$$d : \Omega^k(M) \longrightarrow \Omega^{k+1}(M)$$

determined in coordinates by

$$d\omega = \sum_{i_1 < \cdots < i_k} df_{i_1 \cdots i_k} \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_k}.$$

It obeys the graded product rule

$$d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^{\deg \alpha} \alpha \wedge d\beta \quad \text{and} \quad d^2 = 0.$$

For $f = x^2 y$, $df = 2xy\,dx + x^2\,dy$; and if $\omega = P\,dx + Q\,dy$, then

$$d\omega = \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx \wedge dy.$$

The second formula is the two-dimensional curl expression occurring in Green's theorem. The identity $d^2 = 0$ will also be the reason that every exact form is closed in Section 8.

5 Integration and orientation

An orientation selects which coordinate bases are positive. On an oriented n-manifold, an n-form can be integrated by transporting it to positively oriented coordinate charts, integrating there in the usual multivariable sense, and combining the pieces with a partition of unity. This construction is independent of the chosen oriented atlas. To integrate a form ω over a manifold M, we require $\deg(\omega) = \dim(M)$.

More generally, a k-form is integrated over an oriented k-dimensional manifold or submanifold.

For example, if C is an oriented curve and ω is a 1-form, then $\int_C \omega$ is a line integral. If S is an oriented surface and η is a 2-form, then $\int_S \eta$ is a surface integral. On an oriented three-dimensional region V, a 3-form μ can be integrated as $\int_V \mu$. These examples explain why the degree of a form must agree with the dimension of its domain of integration.

Definition 5.1. The boundary of a manifold with boundary M, denoted ∂M, is the set of points whose local charts map to the boundary of the half-space

$$\mathbb{H}^n = \{(x_1, \ldots, x_n) \in \mathbb{R}^n : x_n \geq 0\}.$$

The boundary is itself an $(n-1)$-dimensional manifold.

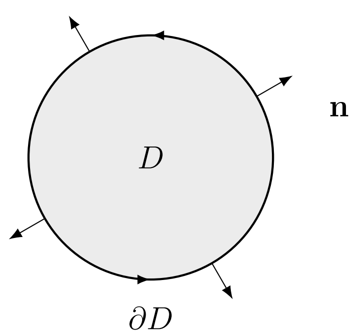


Figure 3: The boundary orientation on ∂D: the outward normal $\mathbf{n}$ followed by the counterclockwise tangent gives the standard orientation of D.

The dimension of the boundary is always one less than the dimension of the original manifold. For example, consider the interval

$$M = [a, b].$$

Its boundary consists of its two endpoints:

$$\partial M = \{a, b\}.$$

The interval is a 1-dimensional manifold, while its boundary is a 0-dimensional manifold. Similarly, the boundary of a disk is its circumference:

$$\partial D^2 = S^1,$$

and the boundary of a solid ball is the sphere

$$\partial B^3 = S^2.$$

In each case, taking the boundary reduces the dimension by one:

$$\dim(\partial M) = \dim(M) - 1.$$

While the boundary operation reduces the dimension of a manifold by one, the exterior derivative increases the degree of a differential form by one:

$$d : \Omega^k(M) \to \Omega^{k+1}(M).$$

These two operations appear to move in opposite directions. However, they are dual operations connected by integration. The generalized Stokes theorem expresses this relationship:

$$\int_M d\omega = \int_{\partial M} \omega.$$

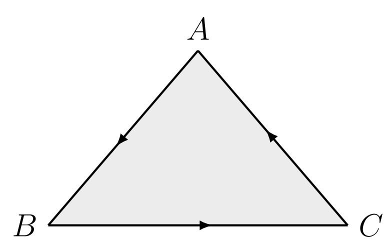
The exterior derivative acts on the form being integrated, while the boundary operation acts on the domain of integration.

Both operations also satisfy a cancellation property:

$$d(d\omega) = 0.$$

Applying the exterior derivative twice always gives zero because a form has no second-order boundary contribution. Similarly, the boundary of a boundary is zero:

$$\partial(\partial M) = 0.$$



$$\partial(\partial\triangle) = (A - B) + (B - C) + (C - A) = 0$$

Figure 4: Each vertex appears once with each sign, so $\partial(\partial M) = 0$. The boundary orientation on each edge is inherited from the counterclockwise orientation of the triangle.

For example, the endpoints of an interval have no boundary of their own. Likewise, the boundary edges of a polygon meet at vertices with opposite orientations, causing the vertices to cancel.

6 The classical theorems in the language of forms

The power of differential forms is not that they replace the familiar vector calculus theorems with unrelated notation. Rather, they put the quantities already used in multivariable calculus into a single system. The following three calculations make that dictionary explicit.

6.1 Green's theorem

Let $D \subset \mathbb{R}^2$ be an oriented region with positively oriented boundary, and let $\mathbf{F} = (P, Q)$ be a smooth planar vector field. Associate to $\mathbf{F}$ the 1-form

$$\alpha = P\,dx + Q\,dy.$$

The exterior derivative is

$$\begin{aligned} d\alpha &= dP \wedge dx + dQ \wedge dy \\ &= (P_x\,dx + P_y\,dy) \wedge dx + (Q_x\,dx + Q_y\,dy) \wedge dy \\ &= (Q_x - P_y)\,dx \wedge dy. \end{aligned}$$

Here $dx \wedge dx = dy \wedge dy = 0$ and $dy \wedge dx = -dx \wedge dy$ explain the minus sign. With the standard orientation of the plane, $dx \wedge dy$ is the usual area element dA. In addition, integrating α around a parametrized boundary curve gives

$$\int_{\partial D} \alpha = \oint_{\partial D} P\,dx + Q\,dy.$$

Therefore generalized Stokes' theorem for the pair (D, α) is exactly

$$\iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA = \oint_{\partial D} P\,dx + Q\,dy,$$

which is Green's theorem. The usual counterclockwise direction is precisely the boundary orientation induced by the standard orientation of $\mathbb{R}^2$.

6.2 Classical Stokes' theorem

Let $S \subset \mathbb{R}^3$ be an oriented surface and write $\mathbf{F} = (P, Q, R)$. This time associate to $\mathbf{F}$ the 1-form

$$\alpha = P\,dx + Q\,dy + R\,dz.$$

The orientation of ∂S is the one for which an outward-pointing vector in the surface, followed by a positively directed tangent vector to ∂S, gives the chosen orientation of S. This is the surface version of the boundary-orientation convention used in generalized Stokes. Computing as above gives

$$\begin{aligned} d\alpha = {} & (R_y - Q_z)\,dy \wedge dz + (P_z - R_x)\,dz \wedge dx \\ & + (Q_x - P_y)\,dx \wedge dy. \end{aligned}$$

The three coefficients are the components of $\nabla \times \mathbf{F}$. To see how the form records the usual surface integral, let $\mathbf{r}(u, v)$ be an orientation-preserving parametrization of S. A direct substitution shows that

$$\mathbf{r}^*(d\alpha) = \big((\nabla \times \mathbf{F})(\mathbf{r}(u, v))\big) \cdot \big(\mathbf{r}_u \times \mathbf{r}_v\big)\, du \wedge dv.$$

Thus $\int_S d\alpha$ is exactly the flux integral of the curl. On a boundary curve $\mathbf{r}(t)$, the pullback of α is

$$\mathbf{r}^*\alpha = \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\, dt,$$

so $\int_{\partial S} \alpha$ is the circulation integral. Generalized Stokes' theorem consequently becomes

$$\iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n}\, dS = \oint_{\partial S} \mathbf{F} \cdot d\mathbf{r}.$$

6.3 The divergence theorem

For the same vector field $\mathbf{F} = (P, Q, R)$, the appropriate object is now the 2-form

$$\beta = P\, dy \wedge dz + Q\, dz \wedge dx + R\, dx \wedge dy.$$

This is the *flux form* of $\mathbf{F}$. Each wedge term measures the component of $\mathbf{F}$ normal to the complementary coordinate plane. For example, $P\, dy \wedge dz$ measures flux through a surface whose normal points in the x-direction. Taking the exterior derivative yields

$$\begin{aligned} d\beta &= P_x\, dx \wedge dy \wedge dz + Q_y\, dy \wedge dz \wedge dx + R_z\, dz \wedge dx \wedge dy \\ &= (P_x + Q_y + R_z)\, dx \wedge dy \wedge dz \\ &= (\nabla \cdot \mathbf{F})\, dV. \end{aligned}$$

On an oriented boundary surface, $\int_{\partial V} \beta$ is the usual outward flux integral. Applying generalized Stokes to (V, β) gives

$$\iiint_V (\nabla \cdot \mathbf{F})\, dV = \iint_{\partial V} \mathbf{F} \cdot \mathbf{n}\, dS,$$

the divergence theorem. The change from a 1-form in classical Stokes to a 2-form here reflects the different quantity being integrated over the boundary: circulation along a curve versus flux across a surface.

7 Generalized Stokes theorem

We now justify Theorem 1.1. The underlying local computation is the Fundamental Theorem of Calculus. For example, on a rectangle $R = [a, b] \times [c, d]$, take $\omega = f(x, y)\, dx$. Then

$$d\omega = -\frac{\partial f}{\partial y}\, dx \wedge dy,$$

so Fubini's theorem and the Fundamental Theorem give

$$\int_R d\omega = \int_a^b \big(f(x,c) - f(x,d)\big)\,dx = \int_{\partial R} \omega.$$

The vertical sides contribute zero because dx vanishes on their tangent vectors; the signs on the horizontal sides are exactly the boundary orientation. This elementary calculation is already a complete instance of generalized Stokes: the derivative has increased the degree from a 1-form to a 2-form, and the 2-dimensional region has a 1-dimensional boundary.

Proof sketch of Theorem 1.1. Choose finitely many oriented boundary charts and interior charts covering the compact manifold M, together with a smooth partition of unity $\{\rho_i\}$ subordinate to that cover. Since $\sum_i \rho_i = 1$,

$$\int_M d\omega = \sum_i \int_M d(\rho_i \omega).$$

In each chart, the preceding Euclidean Stokes computation (in its n- dimensional form) applies to the compactly supported form $\rho_i\omega$. The interior-chart boundary terms vanish, and the boundary-chart terms carry the induced boundary orientation. Hence

$$\sum_i \int_M d(\rho_i\omega) = \sum_i \int_{\partial M} \rho_i \omega = \int_{\partial M} \left(\sum_i \rho_i\right) \omega = \int_{\partial M} \omega.$$

□

This proof reduces the global statement to the Euclidean Stokes theorem in each coordinate chart. The existence of the partition of unity and the Euclidean theorem are standard results; the rectangle computation above shows the local mechanism behind them.

The theorem recovers familiar theorems by choosing n and ω appropriately: $n = 1$ gives the Fundamental Theorem of Calculus; a 1-form on a planar region gives Green's theorem; and suitable 1- and 2-forms in $\mathbb{R}^3$ give the classical Stokes and divergence theorems.

8 de Rham cohomology

Definition 8.1. A form $\omega \in \Omega^k(M)$ is *closed* if $d\omega = 0$ and *exact* if $\omega = d\eta$ for some $\eta \in \Omega^{k-1}(M)$. The *kth de Rham cohomology group* is

$$H^k_{\mathrm{dR}}(M) = \frac{\ker(d : \Omega^k(M) \to \Omega^{k+1}(M))}{\mathrm{im}(d : \Omega^{k-1}(M) \to \Omega^k(M))}.$$

Here we define the kernel of a map T, $\ker(T)$, to be the set of all vectors $v \in V$ that map to 0; $\ker(T) = \{v \in V : T(v) = 0\}$. In de Rham cohomology, the exterior derivative d^k serves as the map while the vector spaces are spaces of k-forms on the manifold $\Omega^k(M)$. Additionally, we define the image of a map $T : V \to W$, $\mathrm{im}(T)$, as the set of all vectors w in the codomain W of T such that for some vector $v \in V$, $T(v) = w$; $\mathrm{im}(T) = \{w \in W : \exists v \in V : T(v) = w\}$.

Again in the context of de Rham cohomology the relevant image for the k-th group is generated by the incoming exterior derivative $d^{k-1} : \Omega^{k-1}(M) \to \Omega^k(M)$.

Because $d^2 = 0$, every exact form is closed, so the quotient is well defined. It records precisely the obstruction to a closed form being a derivative. In other words, the numerator contains all forms with no further exterior derivative, while the denominator identifies those which already arose as an exterior derivative of a lower-degree form.

The quotient allows us to find closed forms that differ by only an exact form. Since exact forms are already exterior derivatives of lower-degree forms, they do not present new topological information. Thus, de Rham cohomology exclusively records the closed forms that cannot be written globally as exterior derivatives.

By the Poincaré Lemma, every closed k-form on $\mathbb{R}^n$ is exact for $k > 0$, so $H^k_{\mathrm{dR}}(\mathbb{R}^n) = 0$ since there are no "holes"; thus there exists a $(k-1)$-form β for every k-form α such that $\alpha = d\beta$ in $\mathbb{R}^n$ for $k > 0$.

Example 8.2 (The punctured plane)**.** On $M = \mathbb{R}^2 \setminus \{(0,0)\}$ let

$$\omega = \frac{-y}{x^2+y^2}\,dx + \frac{x}{x^2+y^2}\,dy.$$

At a point $(x,y) \neq (0,0)$, this form measures infinitesimal change in the polar angle. It is tempting to call it $d\theta$; locally that is correct, but no single smooth angle function θ is defined on the entire punctured plane. The following computation makes the obstruction precise.

Write $\omega = P\,dx + Q\,dy$, where

$$P = \frac{-y}{x^2+y^2} \qquad \text{and} \qquad Q = \frac{x}{x^2+y^2}.$$

Using the coordinate formula for the exterior derivative gives

$$d\omega = \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx \wedge dy.$$

The two derivatives are

$$\frac{\partial Q}{\partial x} = \frac{y^2 - x^2}{(x^2+y^2)^2} \qquad \text{and} \qquad \frac{\partial P}{\partial y} = \frac{y^2 - x^2}{(x^2+y^2)^2}.$$

They agree, so $d\omega = 0$. Thus ω is closed everywhere on M.

To test whether it is exact, integrate around the unit circle

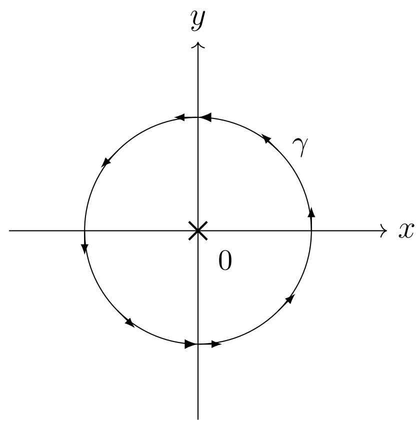
$$\gamma : [0, 2\pi] \longrightarrow M, \qquad \gamma(t) = (\cos t, \sin t).$$

Along this curve, $dx = -\sin t\,dt$, $dy = \cos t\,dt$, and $x^2 + y^2 = 1$. Therefore

$$\begin{aligned} \gamma^*\omega &= (-\sin t)(-\sin t\,dt) + (\cos t)(\cos t\,dt) \\ &= \left(\sin^2 t + \cos^2 t\right) dt = dt. \end{aligned}$$

Consequently,

$$\int_\gamma \omega = \int_0^{2\pi} dt = 2\pi.$$



$$\int_\gamma \omega = 2\pi \neq 0, \quad \text{so } \omega \text{ is closed but not exact.}$$

Figure 5: The angular form $\omega = \frac{-y\,dx + x\,dy}{x^2+y^2}$ on $\mathbb{R}^2 \setminus \{0\}$. Its integral around any loop encircling the origin equals 2π; removing the origin prevents a global potential.

If $\omega = df$ for a globally defined smooth function f on M, then the Fundamental Theorem of Calculus along the closed curve would give

$$\int_\gamma \omega = \int_\gamma df = f(\gamma(2\pi)) - f(\gamma(0)) = 0,$$

because $\gamma(0) = \gamma(2\pi)$. This contradicts the value 2π above, so ω cannot be exact. Therefore $[\omega] \neq 0$ in $H^1_{\mathrm{dR}}(M)$. The nonzero integral measures one trip around the missing origin: the hole creates a closed form that is not the derivative of any globally defined function.

While the two spaces are almost identical, the minute difference of removing a single point creates a nontrivial first cohomology group, illustrating how small topological changes produce dramatic algebraic changes in de Rham cohomology.

Example 8.3. First we recall that a 2-dimensional torus can be written as

$$T^2 = S^1 \times S^1,$$

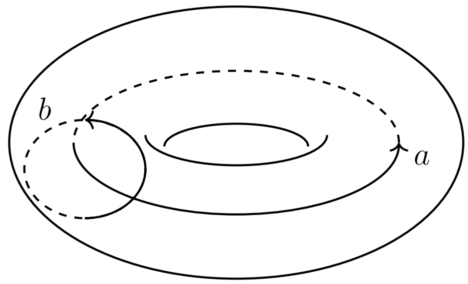
where S^1 is the unit circle, and the Cartesian product consists of all ordered pairs (p, q), such that p and q are points on two copies of S^1.

We can now examine its de Rham cohomology, $H^k_{\mathrm{dR}}(T^2)$ for various k. Starting with $k = 0$, we remember a 0-form is simply a smooth function, $f : T^2 \to \mathbb{R}$. Thus, a closed 0-form satisfies $df = 0$, i.e., every partial derivative is 0. However, we know the only smooth functions with $df = 0$ are constant functions. Thus,

$$\ker(d_0) = \{f \in C^\infty(T^2) \mid \exists\, c \in \mathbb{R} \text{ s.t. } \forall x \in T^2,\ f(x) = c\},$$

which can be written more simply as $\{f : T^2 \to \mathbb{R} \mid f \text{ is constant}\}$. Recall that

$$H^0_{\mathrm{dR}}(T^2) = \frac{\ker(d_0)}{\operatorname{im}(d_{-1})}.$$



$$T^2 = S^1 \times S^1$$

Figure 6: The torus T^2 with longitudinal cycle a (solid) and meridional cycle b (dashed), representing independent generators of $H^1_{\mathrm{dR}}(T^2)$.

Thus, we find the denominator:

$$\mathrm{im}(d_{-1}) = \mathrm{im}(\Omega^{-1}(T^2) \to \Omega^0(T^2)).$$

However, differential forms only exist in nonnegative degrees, so for $k < 0$:

$$\Omega^k(T^2) = 0.$$

Therefore, $\Omega^{-1}(T^2) = 0$ and $\mathrm{im}(d_{-1}) = \{0\}$. Hence:

$$H^0_{\mathrm{dR}}(T^2) = \frac{\{f : T^2 \to \mathbb{R} \mid f \text{ is constant}\}}{\{0\}}.$$

Since the quotient does not identify any nonzero constant functions with each other:

$$H^0_{\mathrm{dR}}(T^2) \cong \mathbb{R}.$$

We now consider $H^1_{\mathrm{dR}}(T^2)$. A general 1-form on the torus can be written locally as

$$\omega = f\,dx + g\,dy.$$

It is closed when $d\omega = 0$. Using the exterior derivative,

$$d\omega = d(f\,dx + g\,dy) = (g_x - f_y)\,dx \wedge dy.$$

Therefore the closed 1-forms satisfy $g_x = f_y$.

The exact 1-forms are those obtained by differentiating smooth functions:

$$\omega = df, \qquad \text{so} \qquad \mathrm{im}(d_0) = \{df : f \in C^\infty(T^2)\}.$$

The first de Rham cohomology group is therefore

$$H^1_{\mathrm{dR}}(T^2) = \frac{\{\text{closed 1-forms}\}}{\{\text{exact 1-forms}\}}.$$

To determine the dimension of this quotient, define the map $I : \Omega^2(T^2) \to \mathbb{R}$ by $I(\omega) = \int_{T^2} \omega$. This map is linear. Moreover, every exact 2-form lies in the kernel of I. Indeed, if $\omega = d\alpha$, then by Stokes' theorem,

$$I(\omega) = \int_{T^2} d\alpha = \int_{\partial T^2} \alpha = 0,$$

because the torus has no boundary. Therefore, $\operatorname{im}(d_1) \subseteq \ker(I)$.

Conversely, if a 2-form has integral zero, then it represents the zero cohomology class. Equivalently, every closed 2-form on T^2 differs from a multiple of the standard area form by an exact form. Thus integration gives an isomorphism

$$H^2_{\mathrm{dR}}(T^2) \cong \mathbb{R}.$$

The single dimension corresponds to the single independent orientation class of the torus.

Although we compute all three groups for the torus, the calculations illustrate a general pattern: H^0 detects connected components, H^1 detects independent loops, and H^2 detects global orientation/area classes.

9 Conclusion

Differential forms make both differentiation and integration intrinsic to a manifold. The exterior derivative satisfies $d^2 = 0$, which produces de Rham cohomology and lets topology appear in calculus. Generalized Stokes' theorem then unifies the major boundary-integral theorems of elementary and multivariable calculus. A natural next step is de Rham's theorem, which identifies de Rham cohomology with singular cohomology with real coefficients.

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