Poincaré–Hopf Theorem

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§1 Introduction

Mathematicians have proved that you cannot comb all the hairs on a hairy ball flat without creating a cowlick, and this is called the Hairy Ball Theorem. Differential topology studies smooth manifolds using local Euclidean coordinates, and analyzes non-trivial spaces using global invariants. This bridges the gap between local linear analysis on Euclidean spaces and global topological constraints. By studying local behavior of smooth functions, we can find new invariant properties of spaces. Local properties like derivatives and tangent spaces can be computed using coordinate charts, but the entire structure of a manifold imposes restrictions on local neighborhoods. An example of this is the behavior of smooth vector fields on manifolds, as many of their properties are constrained by the geometry of manifolds.

This paper investigates the Poincaré–Hopf Theorem and the theorems and invariants leading up to it. This theorem establishes an identity between the sum of local indices of the zeros of a vector field and the Euler characteristic of a compact manifold using local analysis of differential equations to form algebraic invariants. We first define the structure of smooth manifolds, charts, and tangent spaces, which then allows us to define derivatives and critical points rigorously. Using these concepts, we introduce the Inverse Function Theorem, diffeomorphism, and Sard's Theorem, which provide the analytical tools needed to study local behavior and regular values of smooth maps. We can construct a homotopy on these spaces and create an invariant relating preimage of regular values to modulo 2. We then develop smooth homotopies and smooth isotopies, proving the Homotopy Lemma and the Homogeneity Lemma, which establish the invariance properties needed in the study of vector fields and their indices. We next define orientations of manifolds to define the degree of a map, which intuitively measures how many times an oriented manifold wraps around another. This turns out to be another invariant, leading to many topological consequences, notably helpful in the proof of the Poincaré–Hopf Theorem. By applying these topological invariants to smooth vector fields on a manifold, we define the indices of isolated zeros and prove that they are invariant under diffeomorphisms. These results provide the final ingredients needed to establish the Poincaré–Hopf Theorem. We conclude by presenting two applications of the theorem: the Hairy Ball Theorem and the result that every compact odd-dimensional manifold has Euler characteristic zero.

§2 Manifolds

2.1 Smooth Manifolds

A manifold is a topological space that locally resembles a Euclidean space. A **manifold** M is a Hausdorff, second-countable space in which every point p on M admits an open neighborhood $U \subseteq M$ that is homeomorphic to an open subset of $\mathbb{R}^n$ via ϕ. It has dimension n if the Euclidean space it locally resembles is of dimension n.

We can define these local neighborhoods using **charts**, which assign coordinates to points on M. A chart $(U, \phi : U \to \mathbb{R}^n)$ is centered at $p \in U$ if $\phi(p) = 0$.

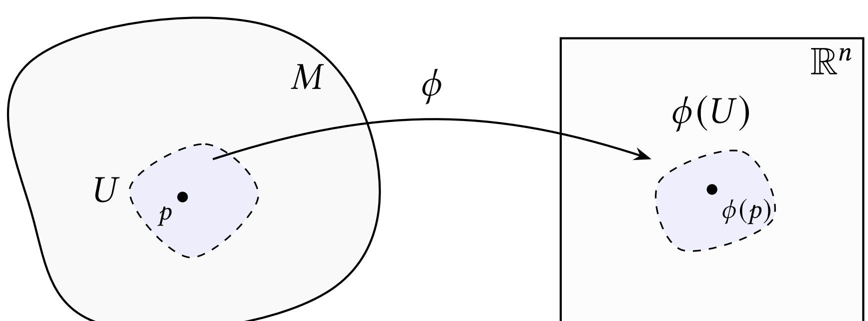

Figure 1: A chart (U, ϕ) mapping an open neighborhood $U \subset M$ onto an open subset $\phi(U) \subset \mathbb{R}^n$.

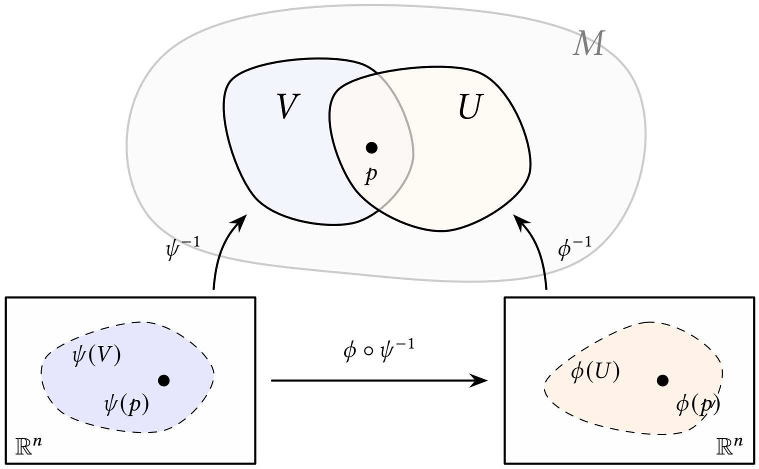

Figure 2: Transition maps between two overlapping charts (U, ϕ) and (V, ψ).

If point p lies in $U \cap V$, the maps (U, ϕ) and (V, ψ) may map p to different coordinates in $\mathbb{R}^n$. The functions that maps the coordinates of $\phi(p)$ to $\psi(p)$ are called **transition maps**. Since these transition maps are composed of homeomorphisms, transition maps are invertible bijections.

Two charts $(U, \phi : U \to \mathbb{R}^n)$ and $(V, \psi : V \to \mathbb{R}^n)$ of a topological manifold M are C^∞-**compatible** (also called compatible) if charts

$$\phi \circ \psi^{-1} : \psi(U \cap V) \to \phi(U \cap V), \quad \psi \circ \phi^{-1} : \phi(U \cap V) \to \psi(U \cap V)$$

are smooth* (infinitely differentiable).

A collection of charts $\{(U_\alpha, \phi_\alpha)\}$ that covers the whole manifold ($M = \bigcup_\alpha U_\alpha$) is called an **atlas**. An atlas is said to be **maximal** if it is not contained in a larger atlas (if $\mathfrak{U}$ contains $\mathfrak{M}$, then $\mathfrak{U} = \mathfrak{M}$). An atlas where every chart in the atlas is smoothly compatible with all other charts in the atlas is called a maximal smooth atlas. A smooth or C^∞ manifold is a topological manifold M equipped with an maximal smooth atlas.

Lemma 1. *Let $\{(U_\alpha, \phi_\alpha)\}$ be an atlas on a locally Euclidean space. If two charts (V, ψ) and (W, σ) are both compatible with the atlas $\{(U_\alpha, \phi_\alpha)\}$, then they are compatible with each other.*

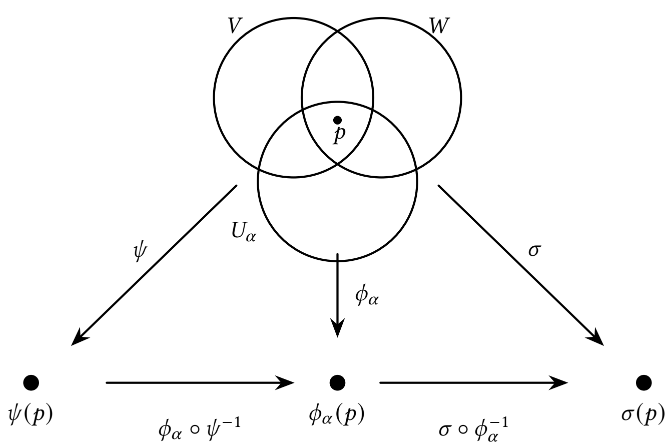

Figure 3: Compatibility of two charts (V, ψ) and (W, σ) via an atlas.

*"smoothly" and C^∞ are used interchangeably

Proof. Let p be an arbitrary point in the intersection $V \cap W$. Since $\{(U_\alpha, \phi_\alpha)\}$ is an atlas, there is some chart containing p such that p lies in the intersection $V \cap W \cap U_\alpha$. Because both (V, ψ) and (W, σ) are compatible with the atlas, the transition maps $\phi_\alpha \circ \psi^{-1}$ and $\sigma \circ \phi_\alpha^{-1}$ are smooth (C^∞). Then we can express the transition map as

$$\sigma \circ \psi^{-1} = (\sigma \circ \phi_\alpha^{-1}) \circ (\phi_\alpha \circ \psi^{-1})$$

Since the composition of smooth functions is smooth, $\sigma \circ \psi^{-1}$ is C^∞ at $\psi(p)$. Because $p \in V \cap W$ was chosen arbitrarily, $\sigma \circ \psi^{-1}$ is smooth on entire $\psi(V \cap W)$. We can also say $\psi \circ \sigma^{-1}$ is smooth on $\sigma(V \cap W)$, meaning (V, ψ) and (W, σ) are smoothly compatible. □

Theorem 2. *Any atlas $\mathfrak{U} = \{(U_\alpha, \phi_\alpha)\}$ on a locally Euclidean space is contained in a unique maximal atlas.*

Proof. Let $\overline{\mathfrak{U}}$ be the set of all charts compatible with the atlas $\mathfrak{U}$. By 1, because any two charts in $\overline{\mathfrak{U}}$ are compatible with the atlas $\mathfrak{U}$, they must be compatible with each other. Thus, $\overline{\mathfrak{U}}$ is a valid atlas, and by definition, it's the unique maximal atlas containing $\mathfrak{U}$. □

Thus, the C^∞ compatibility of charts in an atlas is independent of the choice of charts, thus well-defined.

2.2 Mappings Between Smooth Functions

Once smooth manifolds have been established, we can use charts to extend the notion of differentiability from Euclidean spaces to manifolds. This allows us to define smooth functions and smooth maps independent of the charts chosen. These mappings allow us to do calculus on manifolds.

Smooth maps generalize differentiability from Euclidean spaces to manifolds through local coordinate charts.

Definition 3. A function $f : M \to \mathbb{R}$ on a smooth manifold M is **smooth** at $p \in M$ if there exists a chart (U, ϕ) containing p such that

$$f \circ \phi^{-1} : \phi(U) \subset \mathbb{R}^n \to \mathbb{R}$$

is smooth, independent of the chosen chart.

More generally, let $F : N \to M$ be a continuous map between smooth manifolds. The map F is **smooth** if, for every $p \in N$, there exist charts (U, ϕ) around p and (V, ψ) around $F(p)$ such that

$$\psi \circ F \circ \phi^{-1} : \phi(U \cap F^{-1}(V)) \to \mathbb{R}^m$$

is smooth, independent of the chosen chart.

These properties follow:

- The composition of smooth maps is smooth, and every coordinate map $\phi : U \to \phi(U) \subset \mathbb{R}^n$ is a diffeomorphism[†] onto its image.
- Any diffeomorphism from an open subset of a manifold to an open subset of $\mathbb{R}^n$ defines a valid chart.

We can check smoothness by examining components. If $F = (F_1, \ldots, F_m) : N \to \mathbb{R}^m$, then F is smooth if and only if each component function $F_i : N \to \mathbb{R}$ is smooth. Likewise, for maps between manifolds, we just need to verify that components

[†] A diffeomorphism is a smooth bijection with a smooth inverse

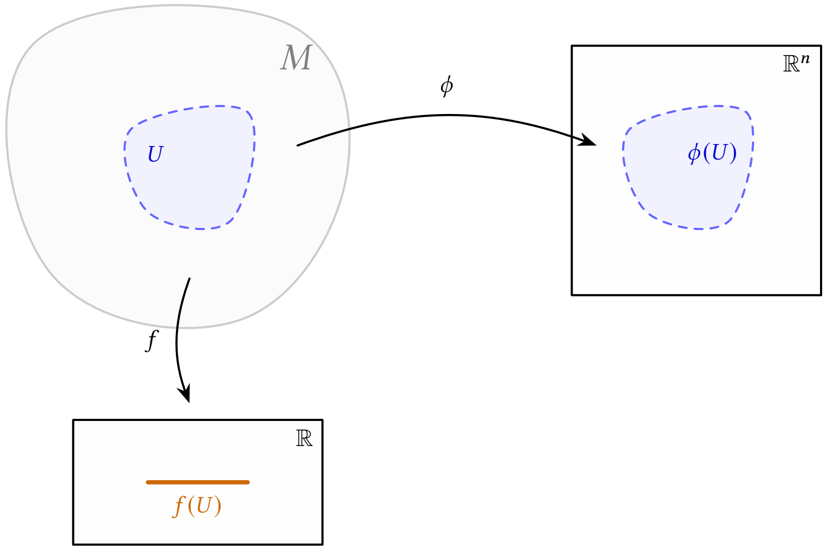

Figure 4: The local coordinate representation $f \circ \phi^{-1}$ of a smooth function $f : M \to \mathbb{R}$.

of F with respect to local charts on the target manifold are smooth.

§3 Tangent Space of Manifolds

3.1 Tangent Spaces and Tangent Vectors

Smooth maps allow us to examine the relationships manifolds, but understanding the geometry of a manifold also requires describing directions and speed. In Euclidean space, these directions are represented by vectors. On a manifold M, however, there is no coordinate system on which vectors can be defined directly. Instead, we can use smooth functions to construct a mapping between a manifold and a Euclidean space at point $p \in M$, leading to the notion of the tangent space at a point.

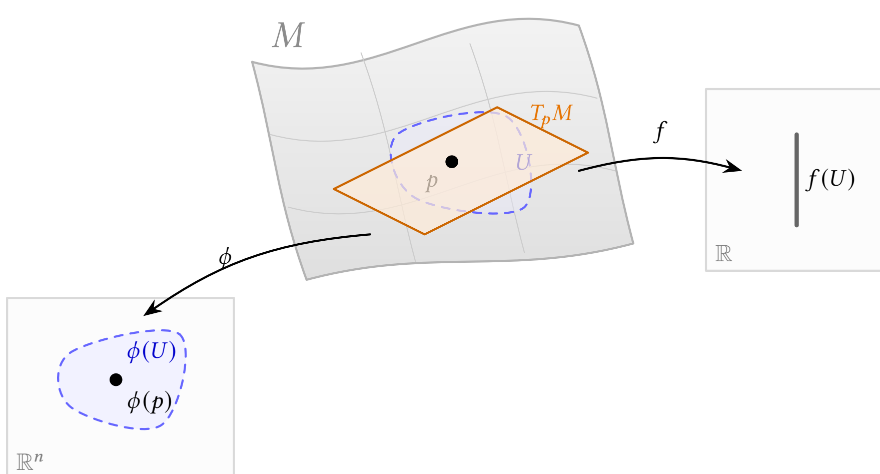

Figure 5: Smooth functions and local charts used to define the tangent space at a point on a manifold.

We define a **germ** of a smooth function at $p \in M$ to be an equivalence class of smooth functions defined in a neighborhood of $p \in M$, two such functions being equivalent if they agree on some smaller neighborhood of p.

A derivation at a point on a manifold M to be a linear map $D : C_p^\infty(M) \to \mathbb{R}$, where $C_p^\infty(M)$ is defined to be the space of smooth functions at point p on M, such that for any $f, g \in C_p^\infty(M)$,

$$D(fg) = D(f)g(p) + f(p)D(g).$$

A tangent vector at a point p in a manifold M is a derivation at p. A vector space T_pM is formed by the tangent vectors at p.

Another way of looking at tangent spaces is through the differential. We first associate with each point x on a smooth manifold $M \subset \mathbb{R}^k$ a linear subspace $T_xM \subset \mathbb{R}^k$ of dimension m called the tangent space of M at x. Then differential df_x will be a linear mapping from T_xM to T_yN, where $y = f(x)$. Elements of T_xM are called tangent vectors to M at x.

3.2 Derivatives

Having introduced smooth manifolds, charts, and smooth functions, we now define derivatives on manifolds. Since a manifold is not, in general, a vector space, derivatives cannot be computed directly as in $\mathbb{R}^n$. Instead, we use local coordinate charts to transfer the problem to Euclidean space, where the usual notion of differentiation applies. This construction provides the foundation for defining g vectors and other differential geometric concepts used later in the paper.

On a manifold M of dimension n, let (U, ϕ) be a chart and f a smooth function on M. As a function into $\mathbb{R}^n$, ϕ has n components $x^1, \ldots, x^n$. This means that if $r^1, \ldots, r^n$ are the standard coordinates on $\mathbb{R}^n$, then $x^i = r^i \circ \phi$. For $p \in U$, we define the **partial derivative** $\frac{\partial f}{\partial x^i}$ of f with respect to x^i at p to be

$$\left.\frac{\partial}{\partial x^i}\right|_p f := \frac{\partial f}{\partial x^i}(p) = \frac{\partial (f \circ \phi^{-1})}{\partial r^i}(\phi(p)) = \left.\frac{\partial}{\partial r^i}\right|_{\phi(p)} (f \circ \phi^{-1}).$$

Since $p = \phi^{-1}(\phi(p))$, this equation may be rewritten in the form

$$\frac{\partial f}{\partial x^i}\big(\phi^{-1}(\phi(p))\big) = \frac{\partial (f \circ \phi^{-1})}{\partial r^i}(\phi(p)).$$

Thus, as functions on $\phi(U)$,

$$\frac{\partial f}{\partial x^i} \circ \phi^{-1} = \frac{\partial (f \circ \phi^{-1})}{\partial r^i}.$$

The partial derivative $\frac{\partial f}{\partial x^i}$ is C^∞ on U because its local representation $\left(\frac{\partial f}{\partial x^i}\right) \circ \phi^{-1}$ is smooth on $\phi(U)$.

3.3 Regular and Critical Points and Values

We now introduce the notions of regular and critical points and values. These concepts are fundamental in differential topology and will be used in the next subsection on Sard's theorem.

Definition 4. Let $F : M \to N$ be a smooth map between smooth manifolds. A point $p \in M$ is a **critical point** of F if the differential

$$dF_p : T_pM \longrightarrow T_{F(p)}N$$

fails to be surjective. Otherwise, p is a **regular point** of F if the differential dF_p is surjective. Equivalently, F is called a **submersion** at p if its differential dF_p is surjective, so p is a regular point of F if and only if F is a submersion at p. In local coordinates, this is equivalent to the Jacobian matrix representing dF_p having full rank. In particular, if $\dim M = \dim N$, then the Jacobian matrix is nonsingular (equivalently, invertible). A point q in N is a **critical value** if it is the image of a critical point; otherwise q is a **regular value**.

Proposition 5. *For a real-valued function $f : M \to \mathbb{R}$, a point p in M is a critical point if and only if relative to some chart $(U, x^1, \ldots, x^n)$ containing p, all the partial derivatives satisfy*

$$\frac{\partial f}{\partial x^j}(p) = 0, \qquad j = 1, \ldots, n.$$

Proof. The differential $df_p : T_pM \longrightarrow T_{f(p)}\mathbb{R} \cong \mathbb{R}$ is represented by the matrix

$$\left[\frac{\partial f}{\partial x^1}(p) \ \cdots \ \frac{\partial f}{\partial x^n}(p)\right].$$

Since the image of df_p is a linear subspace of $\mathbb{R}$, it is either zero- or one-dimensional. In other words, $f_{*,p}$ is either the zero map or a surjective map. Therefore, $f_{*,p}$ fails to be surjective if and only if all the partial derivatives $\frac{\partial f}{\partial x^i}(p)$ are zero. $\square$

3.4 Diffeomorphism

We now introduce diffeomorphisms, which are smooth maps with smooth inverses. They preserve the smooth structure of manifolds and play a fundamental role throughout differential topology.

Proposition 6 (Smoothness of a map in terms of charts)**.** *Let M and N be smooth manifolds, and $F : M \to N$ a continuous map. The following are equivalent:*

(i) The map $F : M \to N$ is smooth.

(ii) There are atlases $\mathfrak{U}$ for M and $\mathfrak{V}$ for N such that for every chart (U, ϕ) in $\mathfrak{U}$ and (V, ψ) in $\mathfrak{V}$, the map

$$\psi \circ F \circ \phi^{-1} : \phi\big(U \cap F^{-1}(V)\big) \longrightarrow \mathbb{R}^m$$

is smooth.

(iii) For every chart (U, ϕ) on M and (V, ψ) on N, the map

$$\psi \circ F \circ \phi^{-1} : \phi\big(U \cap F^{-1}(V)\big) \longrightarrow \mathbb{R}^m$$

is smooth.

A **diffeomorphism** of manifolds is a bijective smooth map $F : M \to N$ whose inverse F^{-1} is also smooth. According to the next two propositions, coordinate maps are diffeomorphisms, and conversely, every diffeomorphism of an open subset of a manifold with an open subset of a Euclidean space can serve as a coordinate map.

Proposition 7. *If (U, ϕ) is a chart on a manifold M of dimension n, then the coordinate map*

$$\phi : U \to \phi(U) \subset \mathbb{R}^n$$

is a diffeomorphism.

Proof. By definition, ϕ is a homeomorphism, so it suffices to check that both ϕ and ϕ^{-1} are smooth. To test the smoothness of $\phi : U \to \phi(U)$, we use the atlas $\{(U, \phi)\}$ with a single chart on U and the atlas $\{(\phi(U), 1_{\phi(U)})\}$ with a single chart on $\phi(U)$. Since

$$1_{\phi(U)} \circ \phi \circ \phi^{-1} : \phi(U) \to \phi(U)$$

is the identity map, it is C^∞. By 6 (ii)⟹(i), ϕ is C^∞.

To test the smoothness of $\phi^{-1} : \phi(U) \to U$, we use the same atlases as above. Since

$$\phi \circ \phi^{-1} \circ 1_{\phi(U)} = 1_{\phi(U)} : \phi(U) \to \phi(U),$$

the map ϕ^{-1} is also C^∞. □

Proposition 8. *Let U be an open subset of a manifold M of dimension n. If $F : U \to F(U) \subset \mathbb{R}^n$ is a diffeomorphism onto an open subset of $\mathbb{R}^n$, then (U, F) is a chart in the differentiable structure of M.*

Proof. For any chart (U_α, ϕ_α) in the maximal atlas of M, both ϕ_α and ϕ_α^{-1} are C^∞ by 7. As compositions of C^∞ maps, both $F \circ \phi_\alpha^{-1}$ and $\phi_\alpha \circ F^{-1}$ are C^∞. Hence, (U, F) is compatible with the maximal atlas. By the maximality of the atlas, the chart (U, F) is in the atlas. □

3.5 Inverse Function Theorem

We now introduce the Inverse Function Theorem, which gives a criterion for when a smooth map is locally invertible. This result will be used later in the paper.

Theorem 9 (Inverse function theorem for $\mathbb{R}^n$)**.** *Let $F : U \to \mathbb{R}^n$ be a C^∞ map defined on an open subset U of $\mathbb{R}^n$. For any point $p \in U$, the map F is locally invertible at p if and only if the Jacobian determinant* $\det\left[\partial F^i / \partial r^j(p)\right]$ *is not zero.*

Theorem 10 (Inverse function theorem for manifolds)**.** *Let $F : M \to N$ be a C^∞ map between two manifolds of the same dimension, and $p \in M$. Suppose for some charts $(U, \phi) = (U, x^1, \dots, x^n)$ about p in M and $(V, \psi) = (V, y^1, \dots, y^n)$ about $F(p)$ in N, $F(U) \subset V$. Set $F^i = y^i \circ F$. Then F is locally invertible at p if and only if its Jacobian determinant* $\det\left[\partial F^i / \partial x^j(p)\right]$ *is nonzero.*

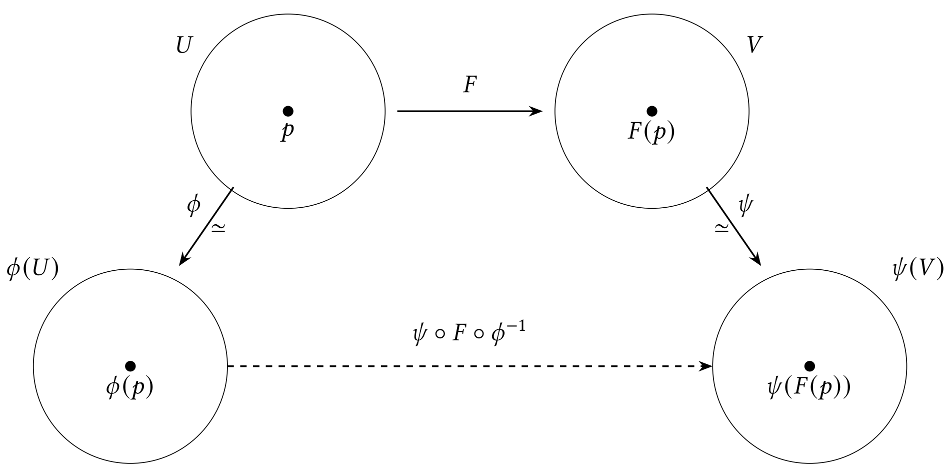

Figure 6: The local coordinate representation $\psi \circ F \circ \phi^{-1}$ of a smooth map $F : M \to N$.

Proof. Since

$$F^i = y^i \circ F = r^i \circ \psi \circ F,$$

the Jacobian matrix of F relative to the charts (U, ϕ) and (V, ψ) is

$$\left[\frac{\partial F^i}{\partial x^j}(p)\right] = \left[\frac{\partial(r^i \circ \psi \circ F)}{\partial x^j}(p)\right] = \left[\frac{\partial(r^i \circ \psi \circ F \circ \phi^{-1})}{\partial r^j}(\phi(p))\right],$$

which is precisely the Jacobian matrix at $\phi(p)$ of the map

$$\psi \circ F \circ \phi^{-1} : \mathbb{R}^n \supset \phi(U) \longrightarrow \psi(V) \subset \mathbb{R}^n$$

between two open subsets of $\mathbb{R}^n$. By the inverse function theorem for $\mathbb{R}^n$,

$$\det\left[\frac{\partial F^i}{\partial x^j}(p)\right] = \det\left[\frac{\partial\big(r^i \circ (\psi \circ F \circ \phi^{-1})\big)}{\partial r^j}(\phi(p))\right] \neq 0$$

if and only if $\psi \circ F \circ \phi^{-1}$ is locally invertible at $\phi(p)$. Since ψ and ϕ are diffeomorphisms (Proposition 7), this last statement is equivalent to the local invertibility of F at p (see 6). □

Proposition 11. Let U be an open subset of a manifold M of dimension n. If $F : U \to F(U) \subset \mathbb{R}^n$ is a diffeomorphism onto an open subset of $\mathbb{R}^n$, then (U, F) is a chart in the differentiable structure of M.

We usually apply the inverse function theorem in the following form.

Corollary 12. *Let N be a manifold of dimension n. A set of n smooth functions $F^1, \dots, F^n$ defined on a coordinate neighborhood $(U, x^1, \dots, x^n)$ of a point $p \in N$ forms a coordinate system about p if and only if the Jacobian determinant $\det\left[\partial F^i/\partial x^j(p)\right]$ is nonzero.*

Proof. Let $F = (F^1, \dots, F^n) : U \to \mathbb{R}^n$. Then, $\det\left[\frac{\partial F^i}{\partial x^j}(p)\right] \neq 0$

$\iff F : U \to \mathbb{R}^n$ is locally invertible at p (by the inverse function theorem)

$\iff$ there is a neighborhood W of p in N such that $F : W \to F(W)$ is a diffeomorphism (by the definition of local invertibility)

$\iff (W, F^1, \dots, F^n)$ is a coordinate chart about p in the differentiable structure of N (by 11). □

3.6 Sard's theorem

We now introduce Sard's theorem, a fundamental result in differential topology. It guarantees the existence of regular values for smooth maps by showing that the set of critical values has Lebesgue measure zero. Since regular values play an important role in the study of smooth maps, we state this theorem as a useful tool for the results that follow.

Theorem 13 (Sard's theorem)**.** *Let $f : U \to \mathbb{R}^n$ be a smooth map, defined on an open set $U \subset \mathbb{R}^m$, and let*

$$C = \{x \in U \mid \operatorname{rank} df_x < n\}.$$

Then the image $f(C) \subset \mathbb{R}^n$ has Lebesgue measure zero.

§4 Homotopy / Degree Modulo 2

4.1 Smooth Homotopies Between Functions

We now introduce smooth homotopies, which describe continuous deformations between smooth maps. This concept allows us to compare smooth maps preserving smoothness and will be used in the arguments developed later in this paper.

Definition 14. Let $X \subset \mathbb{R}^k$ and let $f, g : X \longrightarrow Y$ be smooth maps. We say that f and g are **smoothly homotopic**, denoted by $f \sim g$, if there exists a smooth map

$$F : X \times [0,1] \longrightarrow Y$$

such that

$$F(x,0) = f(x) \quad \text{and} \quad F(x,1) = g(x)$$

for every $x \in X$. The map F is called a **smooth homotopy** from f to g. Intuitively, a smooth homotopy provides a smooth deformation of f into g as the parameter t varies continuously from 0 to 1.

4.2 Smooth Isotopy

We now define smooth isotopies. Unlike a smooth homotopy, a smooth isotopy requires each intermediate map to be a diffeomorphism, making it a stronger notion of deformation.

Definition 15. The diffeomorphism f is **smoothly isotopic** to g if there exists a smooth homotopy $F : X \times [0,1] \longrightarrow Y$ from f to g so that, for each $t \in [0,1]$, the map $x \longmapsto F(x,t)$ is a diffeomorphism between X and Y.

4.3 Homotopy Lemma

We are now ready to present the Homotopy Lemma. One of the central questions in differential topology is which properties of a smooth map remain unchanged under a smooth deformation. The Homotopy Lemma provides an important answer to this question and serves as a fundamental tool for the development of the Poincaré–Hopf Theorem.

Lemma 16 (Homotopy Lemma)**.** *Let $f, g : M \longrightarrow N$ be smoothly homotopic maps between manifolds of the same dimension, where M is compact and without boundary. If $y \in N$ is a regular value for both f and g, then*

$$\#f^{-1}(y) \equiv \#g^{-1}(y) \pmod 2.$$

Proof. Let

$$F : M \times [0,1] \longrightarrow N$$

be a smooth homotopy from f to g.

First assume that y is also a regular value of F. Then $F^{-1}(y)$ is a compact 1-manifold. Its boundary is

$$\partial F^{-1}(y) = F^{-1}(y) \cap \big(M \times \{0\} \,\cup\, M \times \{1\}\big) = f^{-1}(y) \times \{0\} \,\cup\, g^{-1}(y) \times \{1\}.$$

Hence

$$\#\partial F^{-1}(y) = \#f^{-1}(y) + \#g^{-1}(y).$$

A compact 1-manifold has an even number of boundary points. Therefore

$$\#f^{-1}(y) + \#g^{-1}(y)$$

is even, which is equivalent to

$$\#f^{-1}(y) \equiv \#g^{-1}(y) \pmod 2.$$

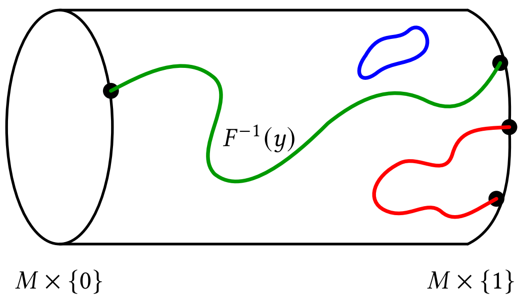

Figure 7: The number of boundary points on the left is congruent to the number on the right modulo 2.

Now suppose that y is not a regular value of F. Since y is a regular value of both f and g, there exist neighborhoods $V_1, V_2 \subset N$ of y such that every point of V_1 is a regular value of f, every point of V_2 is a regular value of g, and

$$\#f^{-1}(y') = \#f^{-1}(y) \qquad \text{for all } y' \in V_1,$$

while

$$\#g^{-1}(y') = \#g^{-1}(y) \qquad \text{for all } y' \in V_2.$$

By Sard's Theorem, the set of regular values of F is dense in N. Since $V_1 \cap V_2$ is a nonempty open neighborhood of y, there exists a regular value $z \in V_1 \cap V_2$ of F. Applying the first part of the proof to z gives

$$\#f^{-1}(z) \equiv \#g^{-1}(z) \pmod 2.$$

Using the equalities above,

$$\#f^{-1}(y) = \#f^{-1}(z) \equiv \#g^{-1}(z) = \#g^{-1}(y) \pmod 2,$$

which proves the result. □

4.4 Homogeneity Lemma

We now present the Homogeneity Lemma, which describes an important property of connected smooth manifolds. It shows that any two interior points can be related by a diffeomorphism that is smoothly isotopic to the identity. This result will be used later to compare the local behavior of vector fields at different points on a manifold.

Lemma 17 (Homogeneity Lemma)**.** *Let y and z be arbitrary interior points of the smooth, connected manifold N. Then there exists a diffeomorphism*

$$h : N \to N$$

that is smoothly isotopic to the identity and carries y into z.

Proof. When $N = S^m$, the result is immediate: a rotation of the sphere carries y to z, is a diffeomorphism, and is smoothly isotopic to the identity.

Now let N be a connected manifold. We first construct a smooth isotopy of $\mathbb{R}^n$ that fixes every point outside the unit ball and moves the origin to any prescribed point of the open unit ball, as illustrated below.

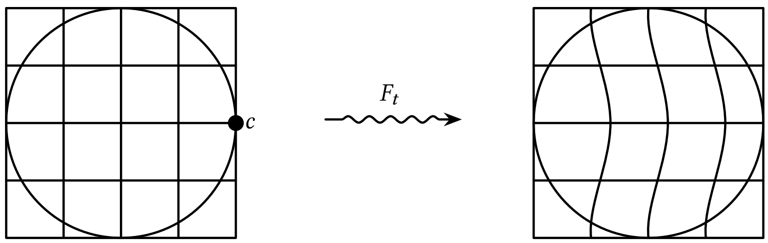

Figure 8: Deforming the unit ball.

Choose a smooth function $\varphi : \mathbb{R}^n \to \mathbb{R}$ satisfying

$$\varphi(x) > 0 \quad \text{for } \|x\| < 1, \qquad \varphi(x) = 0 \quad \text{for } \|x\| \geq 1.$$

For a fixed unit vector $c \in S^{n-1}$, consider the differential equations

$$\frac{dx_i}{dt} = c_i \varphi(x), \qquad i = 1, \ldots, n.$$

The corresponding flow F_t is a smooth family of diffeomorphisms of $\mathbb{R}^n$. Since φ vanishes outside the unit ball, each F_t fixes every point outside the ball. By choosing c and t appropriately, F_t moves the origin to any desired point in the open unit ball.

Every interior point of N has a neighborhood diffeomorphic to an open subset of $\mathbb{R}^n$. Applying the above construction in a coordinate neighborhood and extending it by the identity outside the neighborhood shows that every point sufficiently close to a given interior point is smoothly isotopic to it. Hence every isotopy class of interior points is open.

Since the interior of the connected manifold N is connected, it cannot be partitioned into disjoint nonempty open isotopy classes. Therefore there is only one isotopy class, and for any interior points $y, z \in N$ there exists a diffeomorphism

$$h : N \to N$$

that is smoothly isotopic to the identity and satisfies $h(y) = z$. □

4.5 Degree Modulo 2

Theorem 18. *If y and z are regular values of f then*

$$\#f^{-1}(y) \equiv \#f^{-1}(z) \pmod{2}.$$

This common residue class, which is called the mod 2 degree of f, depends only on the smooth homotopy class of f.

Proof. Given regular values y and z, let h be a diffeomorphism from N to N which is isotopic to the identity and which carries y to z. Then z is a regular value of the composition $h \circ f$. Since $h \circ f$ is homotopic to f, by 16

$$\#(h \circ f)^{-1}(z) \equiv \#f^{-1}(z) \pmod{2}.$$

But

$$(h \circ f)^{-1}(z) = f^{-1}h^{-1}(z) = f^{-1}(y),$$

so that

$$\#(h \circ f)^{-1}(z) = \#f^{-1}(y).$$

Therefore

$$\#f^{-1}(y) \equiv \#f^{-1}(z) \pmod 2,$$

as required.

Call this common residue class $\deg_2(f)$. Now suppose that f is smoothly homotopic to g. By Sard's theorem, there exists an element $y \in N$ which is a regular value for both f and g. The congruence

$$\deg_2 f \equiv \#f^{-1}(y) \equiv \#g^{-1}(y) \equiv \deg_2 g \pmod 2$$

now shows that $\deg_2 f$ is a smooth homotopy invariant. □

§5 Oriented Manifolds and Brouwer Degree

To understand how maps behave between manifolds, we must first know how to define orientations and directions on these spaces. The following section introduces oriented manifolds, which establish rules needed to rigorously define orientation on a manifold.

5.1 Oriented Manifolds

An **orientation** for a finite dimensional real vector space is an equivalence class of ordered bases as follows

- the ordered basis $(b_1, \cdots, b_n)$ determines the same orientation as the basis $(b'_1, ..., b'_n)$ if $b'_i = \sum a_{ij} b_j$ with $\det(a_{ij}) > 0$
- it determines the opposite orientation if $\det(a_{ij}) < 0$.

Thus a finite-dimensional vector space only has 2 orientations.

An **oriented smooth manifold** is a manifold M of dimension m together with a choice of orientation for each tangent space $T_x M$. For each point $p \in M$, there exists an open neighborhood $U \subseteq M$ containing p and a diffeomorphism $h\colon U \to V \subseteq \mathbb{R}^m$. h is said to be orientation-preserving if for every $x \in U$, $dh_x\colon T_x M \to \mathbb{R}^m$ carries the chosen orientation of $T_x M$ to the standard orientation of $\mathbb{R}^m$, which means the determinant of its Jacobian matrix is positive.

If M is connected and orientable, then it has precisely two orientations. If M has a boundary, at boundary point, tangent space vectors are classified as tangent, outward, or inward. A manifold's orientation induces a boundary orientation by placing an outward-pointing vector first in a positive basis. For one-dimensional manifolds, boundary points receive signs based on whether positive vectors point outward or inward.

5.2 Brouwer Degree

Each differential either preserves or reverses orientation, and by summing these signs across all preimages of a regular value, we get an integer that is independent of the choice of regular value. This topological invariant defines the **Brouwer degree**. The Brouwer degree is an integer that measures how many times a smooth map between two oriented manifolds wraps one manifold around the other, with orientation taken into account. It is a fundamental tool in topology which is as an invariant of the map. This tells us whether a map can be continuously deformed into another map. If the degree of a map $f : M \to N$ is nonzero, then the map is surjective, so the equation $f(x) = y$ has a solution $x \in M$ for every $y \in N$.

Definition 19. Suppose M and N are oriented n-dimensional manifolds, where M is compact and N is connected, and let $f : M \to N$ be a smooth map.

For a regular value $y \in N$, every point $x \in f^{-1}(y)$ has an invertible differential

$$df_x : T_xM \to T_yN.$$

Each differential either reserves orientation (+1) or reverses orientation (−1). The degree of f at y is defined by

$$\deg(f; y) = \sum_{x \in f^{-1}(y)} \operatorname{sign}(df_x).$$

We want to show the following theorems:

Theorem 20. *The integer* $\deg(f; y)$ *does not depend on the choice of regular value* y*.*

Theorem 21. *If f is smoothly homotopic to g, then* $\deg f = \deg g$*.*

We first prove the following lemma.

Lemma 22. *Let X be a smooth manifold with boundary $M = \partial X$. If $f : M \to N$ extends to a smooth map $F : X \to N$, then* $\deg(f; y) = 0$ *for every regular value y.*

Proof. First suppose that y is a regular value for F, as well as for $f = F \mid M$. The compact 1-manifold $F^{-1}(y)$ is a finite union of arcs and circles, with only the boundary points of the arcs lying on $M = \partial X$. Let $A \subset F^{-1}(y)$ be one of these arcs, with $\partial A = \{a\} \cup \{b\}$. We will show that

$$\operatorname{sign} df_a + \operatorname{sign} df_b = 0,$$

and hence (summing over all such arcs) that $\deg(f; y) = 0$.

The orientations for X and N determine an orientation for A as follows: Given $x \in A$, let $(v_1, \ldots, v_{n+1})$ be a positively oriented basis for TX_x with v_1 tangent to A. Then v_1 determines the required orientation for TA_x if and only if dF_x carries $(v_2, \ldots, v_{n+1})$ into a positively oriented basis for TN_y.

Let $v_1(x)$ denote the positively oriented unit vector tangent to A at x. Clearly v_1 is a smooth function, and $v_1(x)$ points outward at one boundary point (say b) and inward at the other boundary point a. It follows immediately that

$$\operatorname{sign} df_a = -1, \quad \operatorname{sign} df_b = +1;$$

with sum zero. Adding up over all such arcs A, we have proved that $\deg(f; y) = 0$.

More generally, suppose that y_0 is a regular value for f, but not for F. The function $\deg(f; y)$ is constant within some neighborhood U of y_0. Hence, as in § 4, we can choose a regular value y for F within U and observe that

$$\deg(f; y_0) = \deg(f; y) = 0.$$

□

Now consider a smooth homotopy $F : [0,1] \times M \to N$ between two mappings $f(x) = F(0,x)$, $g(x) = F(1,x)$.

Lemma 23. *The degree* $\deg(g; y)$ *is equal to* $\deg(f; y)$ *for any common regular value* y*.*

Proof. The manifold $[0,1] \times M^n$ can be oriented as a product, and will then have boundary consisting of $1 \times M^n$ (with the correct orientation) and $0 \times M^n$ (with the wrong orientation). Thus the degree of $F\mid_{\partial([0,1]\times M^n)}$ at a regular value y is equal to the difference

$$\deg(g; y) - \deg(f; y).$$

By 22, the difference is 0. □

If y and z are both regular values for $f : M \to N$, choose a diffeomorphism $h : N \to N$ that carries y to z and is isotopic to the identity. Then h will preserve orientation, and $\deg(f; y) = \deg(h \circ f; h(y))$ by inspection. But f is homotopic to $h \circ f$; hence $\deg(h \circ f; z) = \deg(f; z)$ by 23. Therefore $\deg(f; y) = \deg(f; z)$.

Brouwer's degree is also called winding number. We can use Brouwer to show that S^n will have a smooth field of nonzero tangent vectors if and only if n is odd.

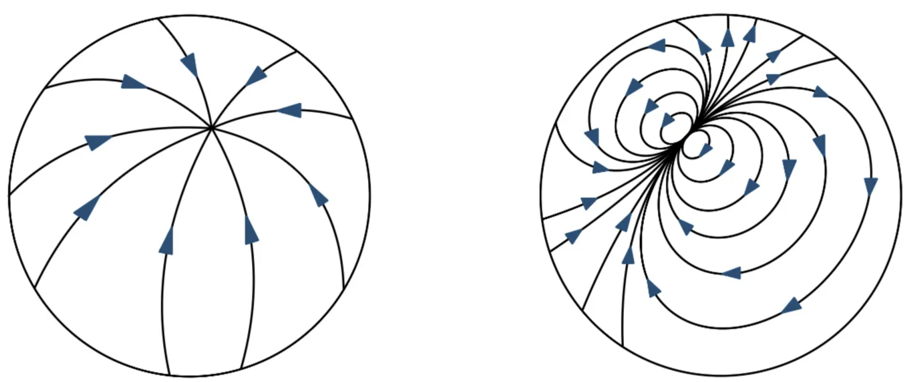

Figure 9: Examples of attempts of smooth fields of nonzero vectors with $\mathbb{S}^2$.

§6 Vector Fields

6.1 Vector Fields

Let M be a smooth manifold. The tangent bundle TM is the disjoint union of all tangent spaces at all points on the manifold. The vector field X is a smooth map $X : M \to TM$. When it projects down to the manifold itself, it must map to the original point, in the sense if $\pi : TM \to M$, then $\pi \circ X = \mathrm{id}_M$.

In a local coordinate chart $(U, x_1, x_2 \dots x_n)$, vector fields can be written as

$$X = \sum_{i=1}^{n} X^i(x) \frac{\partial}{\partial x^i}.$$

The vector field is smooth iff every component function $X^i(x) : U \to \mathbb{R}$ is smooth.

Definition 24. The vector fields v on M and v' on N correspond under f if the derivative df_x carries $v(x)$ into $v'(f(x))$ for each $x \in M$.

If f is a diffeomorphism, then v' is uniquely determined by v and we define $v' = df \circ v \circ f^{-1}$.

Lemma 25. *Suppose that the vector field v on U corresponds to v' on U' under a diffeomorphism $f : U \to U'$. Then the index[1] of v at an isolated zero z is equal to the index of v' at $f(z)$.*

[1]The index is just the Brouwer's degree

A zero in a vector field is a point p on the manifold M such that the velocity at point p is zero. An isolated zero is a zero at point p on the manifold M where there exists an open neighborhood containing p, and contains no other zeros.

6.2 Index of Zeroes

Consider first an open set $U \subset \mathbb{R}^m$ and a smooth vector field

$$v : U \longrightarrow \mathbb{R}^m$$

with an isolated zero at the point $z \in U$. The function

$$\widetilde{v}(x) = \frac{v(x)}{\|v(x)\|}$$

maps a small sphere centered at z into the unit sphere.[‡] The degree of this mapping is called the **index** ι of v at the zero z. Some examples, with indices -1, 0, 1, 2, are illustrated in 10.

[‡]Each sphere is to be oriented as the boundary of the corresponding disk.

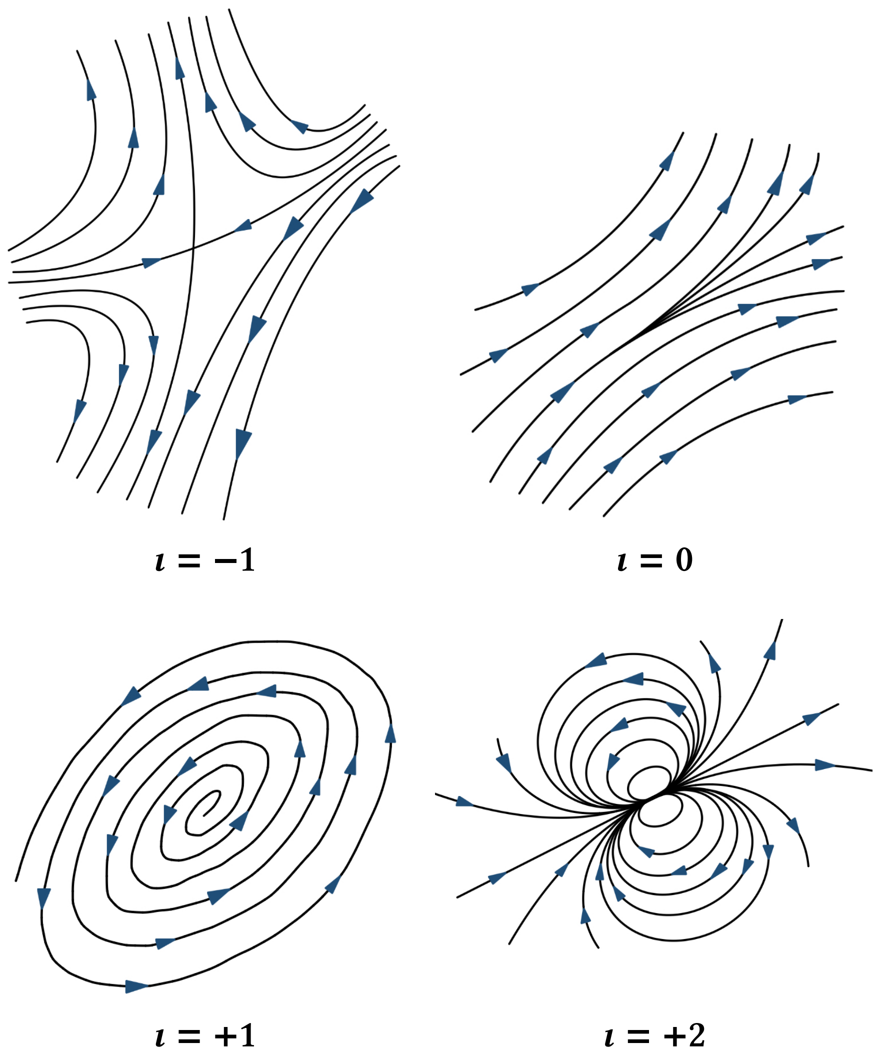

Figure 10: Examples of plane vector fields with isolated zeros of different indices.

6.3 Invariance of index

Lemma 26. *Suppose that the vector field v on U corresponds to*

$$v' = df \circ v \circ f^{-1}$$

on U' under a diffeomorphism $f : U \to U'$. Then the index of v at an isolated zero z is equal to the index of v' at $f(z)$.

Proof. Without loss of generality, assume that

$$z = f(z) = 0,$$

and that U is convex.

First suppose that f preserves orientation. By 27, f is smoothly isotopic to the identity. Hence there exists a smooth family of embeddings

$$f_t : U \to \mathbb{R}^m, \qquad 0 \le t \le 1,$$

such that

$$f_0 = \mathrm{id}, \qquad f_1 = f, \qquad f_t(0) = 0.$$

For each t, define

$$v_t = df_t \circ v \circ f_t^{-1}.$$

Each v_t has an isolated zero at the origin, and for a sufficiently small sphere centered at 0, every v_t is nonvanishing on the sphere. Therefore, the normalized maps

$$\overline{v}_t(x) = \frac{v_t(x)}{\|v_t(x)\|}$$

form a homotopy. Since the degree is invariant under homotopy,

$$\deg(\overline{v}_0) = \deg(\overline{v}_1),$$

which implies

$$\mathrm{ind}_0(v) = \mathrm{ind}_0(v').$$

Now suppose that f reverses orientation. It is enough to consider the case where f is a reflection ρ. Then

$$v' = \rho \circ v \circ \rho^{-1},$$

and the normalized maps satisfy

$$\overline{v}' = \rho \circ \overline{v} \circ \rho^{-1}.$$

Since conjugation by a reflection preserves the degree of the normalized map,

$$\mathrm{ind}_{f(z)}(v') = \mathrm{ind}_z(v).$$

Hence the index is invariant under diffeomorphisms. □

Lemma 27. *Any orientation preserving diffeomorphism f of $\mathbb{R}^m$ is smoothly isotopic to the identity.*

Proof. Assume without loss of generality that $f(0) = 0$. Define

$$F : \mathbb{R}^m \times [0, 1] \longrightarrow \mathbb{R}^m$$

by

$$F(x, t) = \begin{cases} \dfrac{f(tx)}{t}, & 0 < t \leq 1, \\ df_0(x), & t = 0. \end{cases}$$

Since f is smooth, F extends smoothly to $t = 0$, giving a smooth isotopy from f to its derivative df_0.

The derivative df_0 is an orientation-preserving linear isomorphism, and every such linear map is smoothly isotopic to the identity through the connected group $GL^+(m, \mathbb{R})$. Therefore, f is smoothly isotopic to the identity. □

§7 Proof of Poincaré–Hopf Theorem

Theorem 28 (Poincaré–Hopf Theorem). *The sum $\sum \iota$ of the indices at the zeros of such a vector field is equal to the Euler number*[1]

$$\chi(M) = \sum_{i=0}^{m} (-1)^i \text{ rank } H_i(M).$$

It doesn't depend on the particular choice of vector field, and is an invariant of M.

[1] $H_i(M)$ denotes the i-th homology group of M.

Lemma 29 (Hopf Theorem). *If $v : X \longrightarrow R^m$ is a smooth vector field with isolated zeros, and if v points out of X along the boundary, then the index sum $\sum \iota$ is equal to the degree of the Gauss mapping*[1] *$g : \partial X \longrightarrow S^{m-1}$. In particular, $\sum \iota$ does not depend on the choice of v.*

[1] a function that assigns every point on a surface to its corresponding unit normal vector

Proof. By removing an ϵ-ball around each zero, we get a new manifold with boundary. The function $\widetilde{v} = \frac{v(x)}{||v(x)||}$ maps this manifold into S^{m-1}. If a smooth map is defined on a compact manifold with boundary, the total sum of its degrees across all boundary components must be 0. $\widetilde{v}$ to ∂X is homotopic to g, so the degree is just $\deg(g)$. On the inner boundaries, the components add up to $-\Sigma\iota$. Thus $\deg(g) - \Sigma\iota = 0$ is required. □

We can try to compute the index of a vector field v at a zero z in terms of the derivatives of v at z. Consider a vector field v on an open set $U \subset \mathbb{R}^m$ and think of v as a mapping $U \longrightarrow R^m$, so that $dv_z : R^m \longrightarrow R^m$ is defined. The vector field v is **nondegenerate** at z if the linear transformation dv_z is nonsingular (invertible). It follows that z is an isolated zero.

Lemma 30. *The index of v at a nondegenerate zero z is either $+1$ or -1 according as the determinant of dv_z is positive or negative.*

Proof. v is a diffeomorphism from some convex neighborhood U_0 of z into $\mathbb{R}^m$. If v preserves orientation, $v\ |_{U_0}$ can be deformed smoothly into the identity by lemmas 26 and 27. Thus index is $+1$. If v reverses orientation, it can be deformed into the reflection, index is -1. □

Consider a zero z of a vector field w on a manifold $M \subset R^k$. w is a map from M to R^k where derivative $dw_z : TM_z \longrightarrow R^k$ is defined.

Lemma 31. *The derivative dw_z actually carries TM_z into the subspace $TM_z \subset R^k$, and thus can be considered as a linear transformation from TM_z to itself. If this linear transformation has determinant $D \neq 0$ then z is an isolated zero of w with index equal to $+1$ or -1 according as D is positive or negative.*

Proof. Let $h : U \longrightarrow M$ be a parametrization of some neighborhood of z. Let e^i denote the i-th basis vector of R^m and let

$$t^i = dh_u(e^i) = \partial h / \partial u_i$$

so that the vectors $t^1, \cdots, t^m$ form a basis for the tangent space $TM_{h(u)}$.

We want to compute the image of $t_i = t_i(u)$ under the linear transformation $dw_{h(u)}$.

$$dw_{h(u)}(t^i) = d(w \circ h)_u(e^i) = \partial w(h(u))/\partial u_i.$$

. We let $v = \Sigma v_i e^i$ be the vector field on U that corresponds to the vector field w on M. Because $v = dh^{-1} \circ w \circ h$, $w(h(u)) = dh_u(v) = \sum v_i t^i$.. Thus $\partial w(h(u))/\partial u_i = \sum_i (\partial v_i/\partial u_i)t^i + \sum_i v_i(\partial t^i/\partial u_i)$. Evaluating for the zero $h^{-1}(z)$ of v, we obtain

$$dw_z(t^i) = \sum_i (\partial v_i/\partial u_i)t^i.$$

dw_x maps TM_x to itself, the determinant of this linear transformation is equal to the determinant of $(\partial v_i/\partial u_i)$. This with 30, completes the proof. □

Now consider a compact, boundaryless manifold $M \subset \mathbb{R}^k$. Let N_ϵ denote the closed ϵ-neighborhood of M . For ϵ sufficiently small one can show that N_ϵ is a smooth manifold with boundary.

Lemma 32. *For any vector field v on M with only nondegenerate zeros, the index sum $\sum \iota$ is equal to the degree of the Gauss mapping*

$$g : \partial N_\epsilon \to S^{k-1}.$$

In particular this sum does not depend on the choice of vector field.

Proof. For $x \in N_\epsilon$, we let $r(x) \in M$ be the closest point of M. If ϵ is sufficiently small, then the function $r(x)$ is smooth and well defined. The gradient of the squared distance function $\phi(x) = ||x - r(x)||^2$ is grade$\phi = 2(x - r(x))$. Hence, for each point x of the level surface $\partial N_\epsilon = \varphi^{-1}(\epsilon^2)$, the outward unit normal vector is given by

$$g(x) = \text{grad } \varphi/||\text{grad } \varphi|| = (x - r(x))/\epsilon.$$

Extend v to a vector field w on the neighborhood N_ϵ by setting

$$w(x) = (x - r(x)) + v(r(x)).$$

Then w points outward along the boundary since the inner product $w(x)\dot{g}(x)$ is equal to $\epsilon > 0$. Computing the derivative of w at a zero z,

$$\begin{aligned} dw_z(h) &= dv_z(h) \quad \text{for all } h \in TM_z \\ dw_z(h) &= h \qquad \text{for } h \in TM_z^\perp. \end{aligned}$$

Then $\det(dw_z(h)) = \det(dv_z)$. Then the index of w at the zero z is equal to the index ι of v at z. By 29, $\Sigma\iota = \deg(g)$. This completes the proof. □

In order to obtain the full strength of the Poincaré–Hopf Theorem, the following three steps are needed.

7.1 Step 1

Assume M is compact and has no boundary, and that V has only nondegenerate zeros. By 32, $\sigma\iota$ doesn't depend on V. Then we can assume without loss of generate one nondegenerate vector field V for the remaining part of step one. A well known theorem in Morse theory states that every compact smooth manifold admits a Morse function $f : M \to \mathbb{R}$. We can pick a Riemannian metric on M, and let ∇f be its gradient vector field. The zeros of ∇f are the critical points of f. Since f is Morse, all critical points are nondegenerate, and thus we can say that ∇f is a nondegenerate vector field.

Suppose p is a critical point of f. Let $\lambda(p)$ be the Morse index. By the Morse lemma, there are local coordinates centered at p such that

$$f(x_1, \dots, x_m) = f(p) - x_1^2 - \dots - x_{\lambda(p)}^2 + x_{\lambda(p)+1}^2 + \dots + x_m^2.$$

We know

$$\nabla f = (-2x_1, \dots, -2x_{\lambda(p)}, 2x_{\lambda(p)+1}, \dots, 2x_m),$$

so

$$d(\nabla f)_p = \operatorname{diag}(\underbrace{-2, \dots, -2}_{\lambda(p)\text{ entries}}, 2, \dots, 2).$$

Thus

$$\det d(\nabla f)_p = (-2)^{\lambda(p)} 2^{m-\lambda(p)} = (-1)^{\lambda(p)} 2^m.$$

Since p is a nondegenerate zero of ∇f, the index formula for a nondegenerate zero gives

$$\operatorname{ind}_p(\nabla f) = \operatorname{sign}\left(\det d(\nabla f)_p\right).$$

Now,

$$\det d(\nabla f)_p = (-1)^{\lambda(p)} 2^m.$$

Because $2^m > 0$,

$$\operatorname{sign}\left(2^m (-1)^{\lambda(p)}\right) = (-1)^{\lambda(p)}.$$

Therefore,

$$\operatorname{ind}_p(\nabla f) = (-1)^{\lambda(p)}.$$

Since the zeros of ∇f are exactly the critical points of f, we have

$$Z(\nabla f) = \operatorname{Crit}(f).$$

Therefore,

$$\sum_{p \in Z(\nabla f)} \operatorname{ind}_p(\nabla f) = \sum_{p \in \operatorname{Crit}(f)} (-1)^{\lambda(p)}.$$

By Morse theory, we know

$$\chi(M) = \sum_{p \in \operatorname{Crit}(f)} (-1)^{\lambda(p)}.$$

Thus we have

$$\sum_{p \in Z(X)} \operatorname{ind}_p(V) = \chi(M).$$

Theorem 33 (Poincaré–Hopf Theorem 1)**.** *Let M be a compact smooth manifold without boundary, and let V be a smooth vector field on M with nondegenerate zeros.*

$$\sum_{p \in Z(V)} \operatorname{ind}_p(V) = \chi(M).$$

7.2 Step 2

Now to unrestrict this theorem, we let V have isolated degenerate zeros. Because M is compact and $Z(V)$ is discrete, we know the set of zeros is finite. Then we can use the following construction to change the vector field into a nondegenerate one.

Suppose p is an isolated zero. Let $p \subseteq N_1, \overline{N}_1 \subset N$, such that p is the only zero in $\overline{N}$. We define a smooth coordinate map $F : N \longrightarrow \mathbb{R}^m$. Then we define a smooth cutoff function $\lambda : N \longrightarrow [0, 1]$, where $\lambda(x) = 1$ for $x \in N$, and $\lambda(x) = 1$ as x approaches ∂N. By Sard's theorem, regular values are dense. Thus we are able to find a regular value $y \in \mathbb{R}^m$ where $||y||$ is arbitrarily close to 0. Then we can define $V'(x) = V(x) - \lambda(x)y$. Outside of N, the vector field remains unchanged. Inside, we want to verify that there are only nondegenerate zeros.

On N_1, $\lambda = 1$. Thus

$$V'(x) = V(x) - y$$

If $V'(x) = 0$, then $V(x) = y$. Since y is a regular value of V, for every $x \in V^{-1}(y)$, $dV_x : T_xM \longrightarrow \mathbb{R}^m$ is surjective. Both the domain and codomain have dimension of m, thus dV_x is an isomorphism. Because

$$dV'_x = dV_x$$

on N_1, every zero of V' in N_1 is nondegenerate.

We also want to show there are no new zeros in N/N_1. We know $\overline{N/N_1}$ doesn't contain any zeros of V. Then there must exist a $\epsilon > 0$ such that

$$\|V(x)\| \leq \epsilon \text{ for every } x \in \overline{N/N_1}$$

We can pick regular value y such that $\|y\| < \epsilon$. Because $0 \geq \lambda(x) \geq 1$,

$$\|\lambda(x)y\| \leq \|y\| < c \leq \|V(x)\|.$$

Thus

$$V(x) - \lambda(x)y \neq 0$$

on N/N_1. Thus all zeros in V' in N lie in N_1 and all of them can be viewed as nondegenerate.

We also want to show that the index doesn't change when we change isolated zeros to nondegenerate ones because we want to show index remains an invariance under diffeomorphism. When we approach ∂N, $\lambda = 0$, so $V' = V$. That means the normalized map of V and V' are identical on the boundary of N. Then be the definition of index,

$$\operatorname{ind}_p(V) = \deg\left(\frac{V}{\|V\|}\bigg|_{\partial N} \right).$$

The total index of all the zeros of V' inside N has the same boundary degree.

$$\sum_{q \in Z(V') \cap N} \operatorname{ind}_q(V') = \deg\left(\frac{V'}{\|V'\|}\bigg|_{\partial N} \right).$$

We can conclude that

$$\operatorname{ind}_p(V) = \sum_{q \in Z(V') \cap N} \operatorname{ind}_q(V').$$

We are able to change a degenerate zero p into nondegenerate zeros without changing the index. If we apply this to all zeros on V, the Poincaré–Hopf Theorem still holds.

Theorem 34 (Poincaré–Hopf Theorem 2)**.** *Let M be a compact smooth manifold without boundary, and let V be a smooth vector field on M with zeros.*

$$\sum_{p \in Z(V)} \operatorname{ind}_p(V) = \chi(M).$$

The zeros of V can be degenerate or nondegenerate.

7.3 Step 3

Now let M be compact with boundary, with V pointing outward along ∂M. Since V points outward at every boundary point, $V(x) = 0\ \forall x \in \partial M$. That means every zeros of V lies int he interior of M.

Then, we can implement Smooth Collar Theorem. We define a smooth embedding $C : \partial M \times [0, \epsilon) \longrightarrow M$, whose image is a neighborhood of ∂M and $c(x, 0) = x$. We define this embedding to be the collar, and using this collar, we can attach an exterior collar $\partial M \times (\epsilon, 0]$ to M. Define the resulting manifold to be M^+, where M is a submanifold. We can extend V smoothly to M^+ so that all the vectors on the exterior collar point outward and have no zeros. If we were to shrink this collar, no new zeros would appear. Thus $Z(V^+) = Z(V)$. We can conclude

$$\operatorname{ind}_p(V^+) = \operatorname{ind}_p(V).$$

We can apply Step 2 to all isolated zeros on V^+ such that the new vector field V' has only nondegenerate zeros, points outward on the boundary, and the index is unchanged. Then, we have that

$$\sum_{p \in Z(V)} \operatorname{ind}_p(V) = \sum_{q \in Z(V')} \operatorname{ind}_q(V').$$

We now want to show that $\sum_{p \in Z(V)} \operatorname{ind}_p(V) = \chi(M)$. We define a Morse function $f : M \longrightarrow \mathbb{R}$ such that f has no critical points on ∂M, ∇f points outward along ∂M, and all critical points lie in the interior. Just like Step 1, for each critical point p,

$$\operatorname{ind}_p(\nabla f) = (-1)^{\lambda(p)}.$$

Morse cell decomposition for a manifold with boundary gives

$$\chi(M) = \sum_{p \in \operatorname{Crit}(f)} (-1)^{\lambda(p)}.$$

Thus

$$\sum_{p \in Z(\nabla f)} \operatorname{ind}_p(\nabla f) = \chi(M).$$

We can define homotopy

$$H_t = (1 - t)V' + t\nabla f, \quad 0 \le t \le 1.$$

On the boundary of M, both V and ∇f only have positive outward normal components. Thus, all convex combinations H_t also has an only positive outward normal component. In particular $H_t(x) \neq 0$ for $x \in \partial M$. No zero can cross the boundary during the homotopy so by the index invariance,

$$\sum_{q \in Z(V')} \mathrm{ind}_q(\widetilde{V}) = \sum_{p \in Z(\nabla f)} \mathrm{ind}_p(\nabla f) = \chi(M).$$

Therefore,

$$\sum_{p \in Z(V)} \mathrm{ind}_p(V) = \chi(M).$$

This gives us the full version of the Poincaré–Hopf Theorem, as shown below.

Theorem 35 (Poincaré–Hopf Theorem 3)**.** *Let M be a compact smooth manifold, and let V be a smooth vector field on M with zeros. If M has boundary, and V points outward along ∂M. Then*

$$\sum_{p \in Z(V)} \mathrm{ind}_p(V) = \chi(M).$$

The zeros of V can be degenerate or nondegenerate.

§8 Applications of Poincaré–Hopf Theorem

8.1 Hairy Ball Theorem

Theorem 36 (Hairy Ball Theorem)**.** *Every smooth tangent vector field on the 2-dimensional sphere S^2 has at least one zero.*

Proof. Assume, for contradiction, that there exists a smooth tangent vector field X on S^2 that is nowhere zero.

Because X has no zeros, the set of zeros of X is empty. Therefore, the sum of the indices of all zeros of X is the empty sum, so

$$\sum_{p \in Z(X)} \mathrm{ind}_p(X) = 0.$$

On the other hand, the Poincaré–Hopf Theorem states that

$$\sum_{p \in Z(X)} \mathrm{ind}_p(X) = \chi(M).$$

Applying this theorem to $M = S^2$, we obtain

$$\sum_{p \in Z(X)} \mathrm{ind}_p(X) = \chi(S^2).$$

Since

$$\chi(S^2) = 2,$$

it follows that

$$\sum_{p \in Z(X)} \mathrm{ind}_p(X) = 2.$$

This contradicts the earlier conclusion that

$$\sum_{p \in Z(X)} \mathrm{ind}_p(X) = 0.$$

Therefore, no smooth nowhere-vanishing tangent vector field exists on S^2. Hence, every smooth tangent vector field on S^2 has at least one zero. □

8.2 Odd-Dimensional Manifolds

Corollary 37 (Odd-Dimensional Manifolds)**.** *Let M be a compact smooth manifold without boundary. If $\dim M$ is odd, then*

$$\chi(M) = 0.$$

Proof. Let V be a smooth vector field on M with isolated zeros. By the Poincaré–Hopf Theorem,

$$\sum_{p \in Z(V)} \mathrm{ind}_p(V) = \chi(M).$$

Now consider the vector field $-V$. Since V and $-V$ have the same zeros,

$$Z(-V) = Z(V).$$

Moreover, for each $p \in Z(V)$,

$$\mathrm{ind}_p(-V) = (-1)^{\dim M} \mathrm{ind}_p(V).$$

Because $\dim M$ is odd,

$$\mathrm{ind}_p(-V) = -\mathrm{ind}_p(V).$$

Applying the Poincaré–Hopf Theorem to $-V$, we obtain

$$\chi(M) = \sum_{p \in Z(-V)} \mathrm{ind}_p(-V).$$

Therefore,

$$\begin{aligned}\chi(M) &= \sum_{p \in Z(V)} \mathrm{ind}_p(-V) \\ &= -\sum_{p \in Z(V)} \mathrm{ind}_p(V) \\ &= -\chi(M).\end{aligned}$$

Hence,

$$2\chi(M) = 0,$$

and therefore,

$$\chi(M) = 0.$$

□

§9 Bibliography

References

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