

THE PENTAGONAL NUMBER THEOREM

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ABSTRACT. In this note, we discuss Euler’s proof of the pentagonal number theorem and his related work on the subject. We also give Franklin’s combinatorial proof as well as several problems that extend the results discussed in the note.

1. AN INFINITE PRODUCT

In this note, we give an account of Euler’s pentagonal number theorem. For a more detailed account of its history, see [Bel10].

This story starts, at least from a historical viewpoint, with Euler’s consideration of the infinite product

$$(1.1) \quad \prod_{n=1}^{\infty} (1 - x^n).$$

Euler wrote about this product in his paper “Various analytic observations on combinations”¹ (E158 [Eul51b]), written in 1741 and published in 1751. In that paper, he considered various sums and products that can be formed out of an infinite sequence $a_1, a_2, \dots$ of numbers, and one such product he was interested in was the product $\prod_{n=1}^{\infty} (1 - a_n)$, which reduces to (1.1) when we take $a_n = x^n$.

What happens when we expand out (1.1)? We do this by replacing the infinite product with a finite one, say $\prod_{n=1}^N (1 - x^n)$. For small values of N , we get:

$$N = 1 : 1 - x$$

$$N = 2 : 1 - x - x^2 + x^3$$

$$N = 3 : 1 - x - x^2 + x^4 + x^5 - x^6$$

$$N = 4 : 1 - x - x^2 + 2x^5 - x^8 - x^9 + x^{10}$$

$$N = 5 : 1 - x - x^2 + x^5 + x^6 + x^7 - x^8 - x^9 - x^{10} + x^{13} + x^{14} - x^{15}.$$

Eventually, the coefficients stabilize: the coefficient of x^n can never change once we move past the $(1 - x^n)$ term. After doing many more calculations, Euler noticed that

$$\prod_{n=1}^{\infty} (1 - x^n) = 1 - x - x^2 + x^5 + x^7 - x^{12} - x^{15} + x^{22} + x^{26} - x^{35} - x^{40} + x^{51} + \dots.$$

From a quick glance, we can already make a few observations. For instance, all the coefficients are 0 or ± 1 , and the ones that are ± 1 are fairly sparse. They tend to alternate in sign: after the x^0 term, which has a coefficient of $+1$, the next two have coefficients of -1 , then the next two have coefficients of $+1$, and so forth, always two at a time after the initial 1.

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¹Observationes analyticae variae de combinationibus

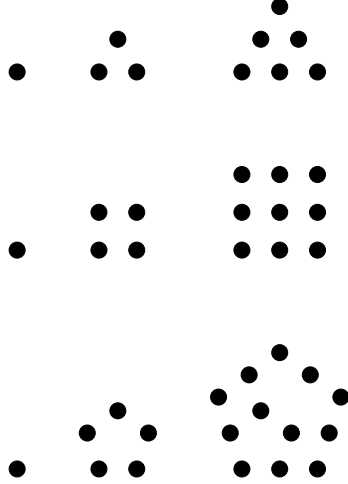


Figure 1. Figurative numbers. Top row: the triangular numbers $1, 3, 6, \dots$, given by $\frac{n^2+n}{2}$. Second row: the square numbers $1, 4, 9, \dots$, given by n^2 . Bottom row: the pentagonal numbers $1, 5, 12, \dots$, given by $\frac{3n^2-n}{2}$.

A slightly subtler problem is to determine precisely which terms have nonzero coefficients. Euler observed that the terms with nonzero coefficient are those of the form $x^{\frac{3n^2 \pm n}{2}}$. Furthermore, those with n even will have coefficient $+1$, whereas those with n odd will have coefficient -1 . In other words, he had observed that

$$(1.2) \quad \prod_{n=1}^{\infty} (1 - x^n) = 1 + \sum_{n=1}^{\infty} (-1)^n \left[x^{\frac{3n^2+n}{2}} + x^{\frac{3n^2-n}{2}} \right].$$

Definition 1.1. A *pentagonal number* is a number of the form $\frac{3n^2 \pm n}{2}$, where n is a nonnegative integer.

As a result, (1.2) is known as the *pentagonal number theorem*. The reason for the name *pentagonal number* is by analogy with triangular numbers, square numbers, and so forth; see Figure 1. The number of dots in a pentagon with n dots to an edge is $\frac{3n^2-n}{2}$. (See problem 3.) If we let n be negative in this expression, which doesn't have a similar geometric interpretation to my knowledge, we get the other "generalized" pentagonal numbers $\frac{3n^2+n}{2}$.

At the time, Euler didn't know how to prove (1.2). That had to wait until later. But he did notice one property of the pentagonal numbers that would come in handy later, when he did prove it. Consider the sequence of positive pentagonal numbers

$$1, 2, 5, 7, 12, 15, 22, 26, 35, 40, 51, \dots$$

If we take the differences between consecutive terms, we get

$$1, 3, 2, 5, 3, 7, 4, 9, 5, 11, \dots$$

It might not be instantly apparent what this sequence is at a glance, but if we take every other term in the sequence starting from the first term, we get the sequence $1, 2, 3, 4, 5, \dots$, whereas if we start with the second term and take every other term, we get the sequence $3, 5, 7, 9, 11, \dots$. Now it's clearer what these successive differences are: half of them are the integers in succession, and half of them are the odd integers in succession, starting from 3.

Indeed, any sequence with these successive differences starting from 1 must be exactly the sequence of pentagonal numbers. (See problem 4.)

2. THE SUM-OF-DIVISORS FUNCTION

When Euler returned to (1.2) next, he still didn't know how to prove it. But he did notice something very surprising that he could do with it, if he assumed it to be true. That was the content of his paper "Discovery of a most extraordinary law of numbers, relating to the sum of their divisors," E175 [Eul51a], written in 1747 and published in 1751. Indeed, it is rather extraordinary.

This is about the sum-of-divisors function $\sigma(n)$. This function is, well, the sum of the divisors of n . For example, $\sigma(12) = 28$ because the divisors of 12 are 1, 2, 3, 4, 6, 12 and $1 + 2 + 3 + 4 + 6 + 12 = 28$. The simplest way of calculating $\sigma(n)$ is in terms of the prime factorization of n : if $n = p_1^{e_1} p_2^{e_2} \cdots p_r^{e_r}$ where the p_i 's are distinct primes, then

$$\sigma(n) = \prod_{i=1}^r (1 + p_i + p_i^2 + \cdots + p_i^{e_i}).$$

For instance, since $12 = 2^2 \cdot 3$, we have

$$\sigma(12) = (1 + 2 + 4)(1 + 3) = 28,$$

as expected. Because it relates to primes, it does not seem as though it ought to be possible to express $\sigma(n)$ in terms of expressions like $\sigma(n-1)$, $\sigma(n-2)$, and so forth, at least not easily. And yet, that is just what Euler managed to do. Here is his remarkable result:

Theorem 2.1. *For every positive integer n , we have*

$$\sigma(n) = \sigma(n-1) + \sigma(n-2) - \sigma(n-5) - \sigma(n-7) + \sigma(n-12) + \sigma(n-15) - \cdots,$$

with two caveats:

- (1) *For any n , at some point the $(n-k)$'s in the expression become negative; if $x < 0$, then we consider $\sigma(x) = 0$.*
- (2) *If we ever encounter the undefined expression $\sigma(0)$ in the computation of $\sigma(n)$, then we replace $\sigma(0)$ with n .*

Example. Let us compute $\sigma(12)$ using this formula. We ought to have

$$\sigma(12) = \sigma(11) + \sigma(10) - \sigma(7) - \sigma(5) + \sigma(0) = 12 + 18 - 8 - 6 + 12 = 28,$$

which is indeed the correct answer.

Remark 2.2. Observe that Theorem 2.1 implicitly contains a primality test. A number n is prime if and only if $\sigma(n) = n + 1$. Thus, we can use Theorem 2.1 to check if $\sigma(n) = n + 1$, and thus whether n is prime. Euler observed this fact in his paper and gave the example of showing that 301 is composite. Note that, while Theorem 2.1 may tell you that a number n is composite, it doesn't tell you what any nontrivial factor of n is. Indeed, as we now know, there are *polynomial-time* (in the number of digits) algorithms for determining if a number is prime, whereas we do not know of a classical polynomial-time algorithm for factoring, and it is widely believed that none exists.

Let us now turn to Euler's proof of Theorem 2.1, at least assuming (1.2). At this point, he was still unable to prove (1.2), so he just assumed it for now. In order to prove Theorem 2.1, Euler started with the purported identity

$$s = (1 - x)(1 - x^2)(1 - x^3) \cdots = 1 - x - x^2 + x^5 + x^7 - x^{12} - x^{15} + \cdots .$$

He then used a popular trick from calculus, especially when dealing with products: the *logarithmic derivative*. The logarithmic derivative of a function $f(x)$ is, by definition, the derivative of its logarithm:

$$\frac{d}{dx} \log(f(x)) = \frac{f'(x)}{f(x)}.$$

Here, we use “log” to denote the natural logarithm, to base e . The logarithmic derivative tends to be very useful when dealing with products, because it converts them into sums. We have

$$\log(s) = \log(1 - x) + \log(1 - x^2) + \log(1 - x^3) + \cdots .$$

After differentiating, we get

$$\frac{s'}{s} = -\frac{1}{1-x} - \frac{2x}{1-x^2} - \frac{3x^2}{1-x^3} - \frac{4x^3}{1-x^4} - \frac{5x^4}{1-x^5} - \cdots .$$

Remark 2.3. At this point, before continuing on, I would like to pause to make a philosophical point about problem-solving. In his paper, Euler wrote out the terms as far as I did above. Euler sometimes did use the Σ notation for sums, but he didn't do so here. Why not? I believe that Σ notation is very useful for writing down finished works concisely. However, it isn't so good for noticing and understanding patterns. By writing out a bunch of terms and aligning them neatly, it was possible for Euler to see exactly what was going on, when it could otherwise have been hidden by overly slick notation.

We can also take the logarithmic derivative of the other expression for s . It isn't particularly pretty though:

$$\frac{s'}{s} = \frac{-1 - 2x + 5x^4 + 7x^6 - 12x^{11} - 15x^{14} + \cdots}{1 - x - x^2 + x^5 + x^7 - x^{12} - x^{15} + \cdots}.$$

It might look prettier if everywhere we multiplied by $-x$, so we get

$$\begin{aligned} -x \frac{s'}{s} &= \frac{x}{1-x} + \frac{2x^2}{1-x^2} + \frac{3x^3}{1-x^3} + \frac{4x^4}{1-x^4} + \frac{5x^5}{1-x^5} + \cdots \\ &= \frac{x + 2x^2 - 5x^5 - 7x^7 + 12x^{12} + 15x^{15} - \cdots}{1 - x - x^2 + x^5 + x^7 - x^{12} - x^{15} + \cdots}. \end{aligned}$$

Let's write t for $-x \frac{s'}{s}$.

Euler next expands out each term in the first expression for t as a sum of a geometric series. Aligning the terms nicely, as Euler does in his paper, we have

$$\begin{array}{cccccccccccccccc}
t = & x & + & x^2 & + & x^3 & + & x^4 & + & x^5 & + & x^6 & + & x^7 & + & x^8 & + & x^9 & + & x^{10} & + & \dots \\
& & & + & 2x^2 & & + & 2x^4 & & & + & 2x^6 & & + & 2x^8 & & + & 2x^{10} & + & \dots \\
& & & & & + & 3x^3 & & & + & 3x^6 & & & + & 3x^9 & & + & \dots \\
& & & & & & + & 4x^4 & & & & + & 4x^8 & & + & 5x^{10} & + & \dots \\
& & & & & & & + & 5x^5 & & & & & + & 6x^6 & & + & \dots \\
& & & & & & & & + & 6x^6 & & & + & 7x^7 & & + & \dots \\
& & & & & & & & & + & 7x^7 & & & + & 8x^8 & & + & \dots \\
& & & & & & & & & & + & 8x^8 & & & + & 9x^9 & & + & \dots \\
& & & & & & & & & & & + & 9x^9 & & & + & 10x^{10} & + & \dots
\end{array}$$

Looking at it this way, it is clear that

$$t = \sum_{n=1}^{\infty} \sigma(n)x^n.$$

Now, it's time to return to the other expression of t :

$$t = \frac{x + 2x^2 - 5x^5 - 7x^7 + 12x^{12} + 15x^{15} - \dots}{1 - x - x^2 + x^5 + x^7 - x^{12} - x^{15} + \dots}.$$

Comparing the two expressions and multiplying both sides by the denominator, we get

$$\sum_{n=1}^{\infty} \sigma(n)x^n (1 - x - x^2 + x^5 + x^7 - x^{12} - x^{15} + \dots) = x + 2x^2 - 5x^5 - 7x^7 + 12x^{12} + 15x^{15} - \dots.$$

To finish up the proof, we just have to compare the coefficients of x^n on the two sides. The coefficient of x^n on the left side is

$$\sigma(n) - \sigma(n-1) - \sigma(n-2) + \sigma(n-5) + \sigma(n-7) - \sigma(n-12) - \sigma(n-15) + \dots,$$

where we truncate the sum at the point where $n - k \leq 0$. The coefficient of x^n on the right side, on the other hand, is $\pm n$ if n is a pentagonal number (with an appropriate sign), and 0 otherwise. That happens precisely when one of the terms on the left side is $\sigma(n - n)$. Thus we have learned that if n is *not* a pentagonal number, then we have

$$\sigma(n) - \sigma(n-1) - \sigma(n-2) + \sigma(n-5) + \sigma(n-7) - \sigma(n-12) - \sigma(n-15) + \dots = 0,$$

which is what Theorem 2.1 claims. On the other hand, when n is a pentagonal number, say with $n = \frac{3k^2 \pm k}{2}$ then we have

$$\sigma(n) - \sigma(n-1) - \sigma(n-2) + \sigma(n-5) + \sigma(n-7) - \dots = (-1)^{k-1}n.$$

This is precisely the correction made by the second caveat in Theorem 2.1. This completes the proof of Theorem 2.1, at least assuming the veracity of (1.2).

3. PROVING THE PENTAGONAL NUMBER THEOREM

Later on, in his paper “A demonstration of a theorem on the order observed in the sum of divisors”², E244 [Eul60], Euler finally managed to prove (1.2). This paper was published in 1760, but he had previously written of his proof in a 1750 letter to Goldbach. As in his proof of Theorem 2.1, this argument is a classic example of Euler’s incredible skill in algebraic manipulations. Let’s have a look!

²Demonstratio theorematism circa ordinem in summis divisorum observatum

Recall that Euler wanted to evaluate the product

$$s = (1-x)(1-x^2)(1-x^3)(1-x^4)(1-x^5)\cdots.$$

He did this by partially expanding it out in a curious way:

$$s = 1 - x - x^2(1-x) - x^3(1-x)(1-x^2) - x^4(1-x)(1-x^2)(1-x^3) - \cdots.$$

Since all the terms in this expansion of s except the $1-x$ are divisible by x^2 , Euler writes

$$s = 1 - x - Ax^2,$$

where

$$A = 1 - x + x(1-x)(1-x^2) + x^2(1-x)(1-x^2)(1-x^3) + x^3(1-x)(1-x^2)(1-x^3)(1-x^4) + \cdots.$$

Euler then expanded out the $1-x$ parts of each factor, and nothing else, which he wrote in an organized manner as

$$\begin{array}{ccccccc} A & = & -x & & -x^2(1-x^2) & & -x^3(1-x^2)(1-x^3) & & -\cdots \\ & & 1 & +x(1-x^2) & +x^2(1-x^2)(1-x^3) & +x^3(1-x^2)(1-x^3)(1-x^4) & +\cdots \end{array}$$

Adding the columns, we get

$$A = 1 - x^3 - x^5(1-x^2) - x^7(1-x^2)(1-x^3) - x^9(1-x^2)(1-x^3)(1-x^4) - \cdots.$$

Now it's time to repeat the trick. Every term except the $1-x^3$ is divisible by x^5 , so Euler writes $A = 1 - x^3 - Bx^5$, where

$$B = 1 - x^2 + x^2(1-x^2)(1-x^3) + x^4(1-x^2)(1-x^3)(1-x^4) + \cdots.$$

Again, he repeats the idea from before, expanding out the $1-x^2$ from each term, so as to give

$$\begin{array}{ccccccc} B & = & -x^2 & & -x^4(1-x^3) & & -x^6(1-x^3)(1-x^4) & & -\cdots \\ & & 1 & +x^2(1-x^3) & +x^4(1-x^3)(1-x^4) & +x^6(1-x^3)(1-x^4)(1-x^5) & +\cdots \end{array}$$

Adding the columns gives

$$B = 1 - x^5 - x^8(1-x^3) - x^{11}(1-x^3)(1-x^4) - x^{14}(1-x^3)(1-x^4)(1-x^5) - \cdots.$$

Time to use the same trick yet again: we have $B = 1 - x^5 - Cx^8$, where

$$C = 1 - x^3 + x^3(1-x^3)(1-x^4) + x^6(1-x^3)(1-x^4)(1-x^5) + \cdots,$$

or

$$\begin{array}{ccccccc} C & = & -x^3 & & -x^6(1-x^4) & & -x^9(1-x^4)(1-x^5) & & -\cdots \\ & & 1 & +x^3(1-x^4) & +x^6(1-x^4)(1-x^5) & +x^9(1-x^4)(1-x^5)(1-x^6) & +\cdots \end{array}$$

or

$$C = 1 - x^7 - x^{11}(1-x^4) - x^{15}(1-x^4)(1-x^5) - x^{19}(1-x^4)(1-x^5)(1-x^6) - \cdots.$$

Using the same trick again with $C = 1 - x^7 - Dx^{11}$, he deduces similarly that

$$D = 1 - x^9 - x^{14}(1-x^5) - x^{19}(1-x^5)(1-x^6) - x^{24}(1-x^5)(1-x^6)(1-x^7) - \cdots.$$

Similarly, we can write $D = 1 - x^9 - Ex^{14}$, then $E = 1 - x^{11} - Fx^{17}$, and so forth.

Let's stand back and see what we have learned:

$$\begin{aligned} s &= 1 - x - Ax^2 \\ Ax^2 &= x^2(1 - x^3) - Bx^7 \\ Bx^7 &= x^7(1 - x^5) - Cx^{15} \\ Cx^{15} &= x^{15}(1 - x^7) - Dx^{26} \\ Dx^{26} &= x^{26}(1 - x^9) - Ex^{40}, \end{aligned}$$

and so on. Putting all this together, we get

$$s = 1 - x - x^2 + x^5 + x^7 - x^{12} - x^{15} + x^{22} + x^{26} - x^{35} - x^{40} + x^{51} + \dots$$

From here, it is easy to check that the successive differences in the exponents are what they are supposed to be for the sequence of pentagonal numbers, and with the desired signs; you will prove this in problem 4.

4. THE PARTITION FUNCTION

The pentagonal number theorem is relevant to the partition function as well as the sum-of-divisors function. A *partition* of a nonnegative integer n is a way of representing n as a sum of positive integers, where order doesn't matter. For example, $3 + 1 + 1$ is a partition of 5, and it counts as the same partition as $1 + 3 + 1$ or $1 + 1 + 3$. We write $p(n)$ for the number of partitions of n . Note that $p(0) = 1$, and we set $p(n) = 0$ for $n < 0$.

The most fruitful way of studying the partition function is via its generating function $P(x) = \sum_{n=0}^{\infty} p(n)x^n$. Note that we previously worked out the generating function for the sum-of-divisors function; that was our function t above. What can we do with the partition function?

Each partition of n contains some number (at least 0) of 1's, some number of 2's, some number of 3's, and so forth. The way we represent the possible number of 1's showing up in a partition is as $1 + x + x^2 + x^3 + \dots$. The 1, or x^0 , is related to partitions with no 1's. The x , or x^1 , is related to the partitions with exactly one 1, and so forth. This series is an infinite geometric series, summing to $\frac{1}{1-x}$.

We can do the same trick with the 2's, giving us the series $1 + x^2 + x^4 + x^6 + x^8 + \dots = \frac{1}{1-x^2}$. If we multiply these two generating functions $\frac{1}{1-x} \cdot \frac{1}{1-x^2}$, we get the generating function for the number of partitions using only 1's and 2's. If we wish to include 3's as well, we multiply by $\frac{1}{1-x^3}$, and so forth. If we allow arbitrary parts in our partition, then we get the full generating function

$$P(x) = \sum_{n=0}^{\infty} p(n)x^n = \prod_{n=1}^{\infty} \frac{1}{1-x^n}.$$

Now it's possible to use the same trick as we used for the σ function to get a recurrence for $p(n)$. If we clear denominators, we get

$$(4.1) \quad \sum_{n=0}^{\infty} p(n)x^n(1-x)(1-x^2)(1-x^3)(1-x^4)\dots = 1.$$

We recognize this product $(1-x)(1-x^2)(1-x^3)(1-x^4)\dots$ from the pentagonal number theorem: it sums to

$$(1-x)(1-x^2)(1-x^3)(1-x^4)\dots = 1 - x - x^2 + x^5 + x^7 - x^{12} - x^{15} + \dots$$

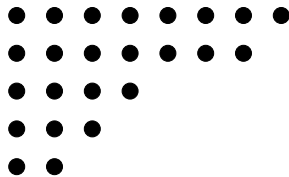


Figure 2. The Ferrers diagram for the partition $24 = 8 + 7 + 4 + 3 + 2$.

When we compare coefficients of x^n in (4.1), we find that $p(0) = 1$ (which is correct), and

$$(4.2) \quad p(n) - p(n-1) - p(n-2) + p(n-5) + p(n-7) - p(n-12) - p(n-15) + \cdots = 0$$

whenever $n \geq 1$. In other words, the partition function satisfies *almost* the same recurrence as does the sum-of-divisors function. Who would have expected that?

5. A BIJECTIVE PROOF OF THE PENTAGONAL NUMBER THEOREM

Euler's proof of the Pentagonal Number Theorem involved clever algebraic manipulations and summing and unsumming series in creative ways, something he was spectacularly good at. Nowadays, we know of several other proofs. Here is one that comes from combinatorics, using partitions. This proof is due to Fabian Franklin [Fra81].

We saw that the generating function for the full partition function is $\prod_{n=1}^{\infty} \frac{1}{1-x^n}$. We might wonder what the reciprocal $\prod_{n=1}^{\infty} (1-x^n)$ counts. To see this, let's start by looking at the related generating function $\prod_{n=1}^{\infty} (1+x^n)$, which is the same but with positive signs instead of negative signs. This is the generating function for $p_d(n)$, the number of partitions of n such that all the parts are distinct, for the same reason as before.

What happens when we insert the minus signs now? This time, it's counting the number of partitions of n into distinct parts, except the partitions of n into an *even* number of distinct parts are counted as $+1$, whereas the partitions of n into an *odd* number of distinct parts are counted as -1 . Thus the coefficient of x^k in $\prod_{n=1}^{\infty} (1-x^n)$ is equal to the number of partitions of k into an even number of distinct parts, minus the number of partitions of k into an odd number of distinct parts. The pentagonal number theorem says that this number is 0, unless k is a pentagonal number, in which case it's ± 1 .

Thus it makes sense to try to pair the partitions of k into distinct parts, where each pair contains one partition with an even number of parts and one has an odd number of parts. The idea is that we should get a complete pairing when k is not a pentagonal number, whereas we should have one unpaired partition when k is a pentagonal number.

To do this, it's helpful to visualize a partition using a *Ferrers* diagram. Suppose we have a partition $k = \lambda_1 + \lambda_2 + \cdots + \lambda_r$, where $\lambda_1 > \lambda_2 > \cdots > \lambda_r$. (The inequalities need not be strict, but we'll only deal with partitions with distinct parts here.) Then we draw r rows of dots, with λ_1 in the first row, λ_2 in the second row, and so forth. See Figure 2 for an example.

To form our pairing, we look at two things in our Ferrers diagram. First, we start with the top right dot, and then we move diagonally down and to the left for as long as we can until there are no dots there. We call the number of dots in that diagonal row s . Then we let m be the number of dots in the bottom row. See Figure 3 for an example, with s dots colored red and m dots colored blue.

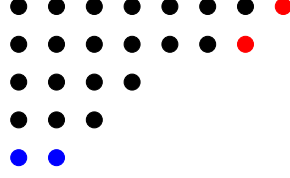


Figure 3. $s = 2$ dots colored red, and $m = 2$ dots colored blue.

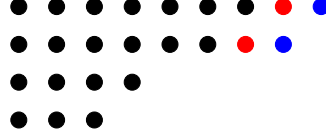


Figure 4. Creating a new diagonal for the blue dots.

Now, if $s < m$, then we can move the red dots to a new row below the blue dots. This generates a new partition of the same number k with distinct parts, but if the old partition had an odd number of parts, then the new one has an even number, and vice versa.

On the other hand, if $s \geq m$, then we move the m blue dots to a new diagonal starting to the right of the top right one. In our example, $s = m = 2$, so this is what we do. See Figure 4 for the result. Here too, we create a valid partition of k into distinct parts, and if the old partition had an odd number of parts, then the new one has an even number, and vice versa.

Note furthermore that these two operations are inverses, so if we start with a partition where $s \geq m$, the resulting partition has $s < m$ and reverses the process, and vice versa.

However, this doesn't quite always work. What can go wrong? There are two things that can go wrong:

- (1) $s = m$, and the right diagonal meets the bottom row, i.e. there is a dot that is supposed to be colored both red and blue. When that happens and we try to perform the above process, then we end up with something like what happens in Figure 5, where the bottom row is aligned wrong. When this happens, we have m parts with smallest part m , and the parts are consecutive, so we must have

$$k = m + (m + 1) + \cdots + (m + (m - 1)) = \frac{m(3m - 1)}{2},$$

which is indeed a pentagonal number.

- (2) $m = s + 1$, and the right diagonal meets the bottom row. In this case, after we try to apply the procedure, we no longer have distinct parts, as shown in Figure 6. In this case, we have s consecutive consecutive parts with smallest part $s + 1$, so we must have

$$k = (s + 1) + (s + 2) + \cdots + (s + 1 + (s - 1)) = \frac{s(3s + 1)}{2},$$

which is also a pentagonal number.

We have shown that we get a complete pairing when k is not a pentagonal number, so the coefficient of x^k in $\prod_{n=1}^{\infty} (1 - x^n)$ is 0. When k is a pentagonal number, we get an almost-complete pairing, with only one partition unpaired. The exceptional partitions of $\frac{r(3r-1)}{2}$ and



Figure 5. A failed attempt at pairing.



Figure 6. Another failed attempt at pairing.

$\frac{r(3r+1)}{2}$ for $r \geq 1$ each have exactly r parts, so each contributes $(-1)^r$. Finally, the empty partition contributes the constant term 1.

Another consequence, which is not obvious on its own, is that $p_d(n)$ is even if n is not a pentagonal number, and $p_d(n)$ is odd if n is a pentagonal number. By contrast, the distribution of even and odd values of $p(n)$ is much less well understood. It is conjectured (see [PS67]) that $p(n)$ is even for asymptotically half of the nonnegative integers n and odd for the other half, but the best results known at present are far weaker than this; see for instance [OS25].

6. PROBLEMS

- (1) Use Theorem 2.1 to compute $\sigma(15)$ and $\sigma(20)$, assuming that you already know all smaller values. Verify that you do indeed get the correct answers.
- (2) Compute a table of $p(n)$ for $n \leq 30$.
- (3) (a) Prove that the number of dots in the pentagons as shown in Figure 1 is, in fact, $\frac{3n^2-n}{2}$.
 (b) More generally, show that the number of dots in the k -gonal analogue of pentagonal numbers is $\frac{n^2(k-2)-n(k-4)}{2}$.
 (c) Show that every hexagonal number is also a triangular number.
- (4) Consider the sequence of numbers starting with 1, such that the difference between consecutive terms is 1, 3, 2, 5, 3, 7, 4, 9, 5, 11, and so forth. The pattern is that every other term in the sequence is 1, 2, 3, 4, 5, and so forth, while the other half are 3, 5, 7, 9, 11, and so forth. Prove that this sequence of numbers is, in fact, the pentagonal numbers.
- (5) As we saw in problem 3, the k -gonal numbers are $\frac{n^2(k-2)-n(k-4)}{2}$. If we allow n to be negative, then we ought to include numbers of the form $\frac{n^2(k-2)+n(k-4)}{2}$ as (generalized) k -gonal numbers. What are the sequences of differences between consecutive k -gonal numbers? For pentagonal numbers, you answered this in problem 4. For triangular numbers and square numbers, respectively, the differences are the consecutive integers and the consecutive odd integers. Give a complete answer in terms of k .
- (6) Prove that for any integer $n \geq 1$, we have

$$p(n) = \frac{1}{n} \sum_{k=1}^n \sigma(k)p(n-k).$$

- (7) Let $p_o(n)$ be the number of partitions of n into odd parts, and let $p_d(n)$ be the number of partitions of n into distinct parts. Show that $p_o(n) = p_d(n)$ for all n .
- (8) Express $\sum_{n=0}^{\infty} p_o(n)x^n$ in a concise form, as an infinite product.
- (9) Find a recurrence similar to (4.2) for $p_o(n)$ (or $p_d(n)$).
- (10) Let $p(n, k)$ denote the number of partitions of n into exactly k parts. Show that $p(n, k) = p(n-1, k-1) + p(n-k, k)$.
- (11) Given a partition λ of n , define its conjugate partition λ^c to be the partition obtained by interchanging the rows and columns of the Ferrers diagram of λ . For instance, if λ is the partition $7+3+1$, then λ^c is the partition $3+2+2+1+1+1+1$. A partition λ is said to be *self-conjugate* if $\lambda = \lambda^c$. Show that the number of self-conjugate partitions of n is equal to the number of partitions of n into odd distinct parts.
- (12) Consider the *Lambert series*

$$f(x) = \sum_{n=1}^{\infty} \frac{x^n}{1-x^n}.$$

Suppose we expand $f(x)$ as a power series $\sum_{n=0}^{\infty} a_n x^n$. Describe the coefficients a_n , and give a proof for your formula.

- (13) Let $a_1, a_2, \dots$ be any sequence of (not necessarily positive) integers. Show that there exist nonnegative integers $m_1, m_2, \dots$ and $\varepsilon_1, \varepsilon_2, \dots \in \{1, -1\}$ such that

$$\prod_{k=1}^{\infty} (1 + \varepsilon_k x^k)^{m_k} = 1 + \sum_{n=1}^{\infty} a_n x^n.$$

(For instance, if all the ε_k 's are equal to -1 and all the m_k 's are equal to 1 , we get the pentagonal number formula.)

- (14) (a) Find all the ε_k 's and m_k 's in problem 13 when $a_n = 1$ for all n .
 (b) Do the same when $a_n = 2$ for all n .
 (c) What about when $a_n = c$ for all n , where c is some positive integer?
- (15) Identify the numbers a_n defined by

$$\prod_{k=1}^{\infty} (1 - x^k)^3 = \sum_{n=0}^{\infty} a_n x^n.$$

What are the coefficients, and what are the exponents with nonzero coefficients? First experiment to guess the answer, and then prove it.

- (16) Prove that

$$\sum_{n=0}^{\infty} p(n)x^n = \prod_{n=1}^{\infty} \frac{1}{1-x^n} = \sum_{n=0}^{\infty} \frac{x^{n^2}}{(1-x)^2(1-x^2)^2 \cdots (1-x^n)^2},$$

where the $n=0$ summand on the right is interpreted as 1 . (Hint: Given a partition of n , look at its Ferrers diagram and find the largest square in it. This square is known as the *Durfee square*.)

- (17) Let $\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$ be the Riemann zeta function. Show that

$$\sum_{n=1}^{\infty} \frac{\sigma(n)}{n^s} = \zeta(s)\zeta(s-1)$$

as formal Dirichlet series. More generally, if $\sigma_k(n)$ is the sum of the k^{th} powers of the divisors of n , i.e. $\sigma_k(n) = \sum_{d|n} d^k$, show that

$$\sum_{n=1}^{\infty} \frac{\sigma_k(n)}{n^s} = \zeta(s)\zeta(s-k).$$

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