

HILBERT'S THIRD PROBLEM

SIMON RUBINSTEIN-SALZEDO

ABSTRACT. Hilbert's Third Problem asks whether, given two polyhedra of the same volume, it is possible to cut one into finitely many polyhedra and rearrange the pieces into the other one. In these notes, we will answer this question and study the surrounding mathematics. The notes conclude with a selection of problems on Hilbert's Third Problem and related topics.

1. HILBERT'S 1900 ADDRESS

Surely the most famous talk in the history of mathematics was David Hilbert's address at the 1900 International Congress of Mathematicians. Generally at a talk, mathematicians discuss work they have done. Less commonly, we talk about work that others have done. But it is much less common to talk about work that hasn't been done.

In his talk at the 1900 ICM, Hilbert broke strongly with tradition and described his vision for mathematics in the next century. His vision was that the mathematics to be developed would be shaped by the most important problems that were then still open. So, in his talk, he described ten of what he considered the most important open problems in mathematics. In his corresponding paper, he extended the list to twenty-three problems. Many of these problems are still known by their Hilbert number today. For example, Hilbert's Tenth Problem is to determine whether there is an algorithm to determine whether a Diophantine equation has an integer solution. Hilbert's Third Problem is another one of them that is well-known by its Hilbert number. This one was the first of Hilbert's problems to be solved, by Max Dehn. See [Hil02] for Hilbert's original list of problems.

2. DISSECTING POLYGONS

Before we state Hilbert's Third Problem, let us look at a simpler version of it.

Question 2.1. *Let P and Q be two polygons. Is it possible to cut P up into finitely many polygons and rearrange them into Q ?*

Here and throughout these notes, by a “polygon” (or a polyhedron, etc.), we mean the filled-in figure, including both the boundary and the interior.

Definition 2.2. If it is possible to cut P up into finitely many polygons and rearrange them into Q , then we say that P and Q are *equidecomposable*.

Remark 2.3. Another common term is *scissors congruent*.

See Figure 1 for an example of such a decomposition.

Remark 2.4. It is sometimes convenient to require that we cut up P into triangles. This does not change anything, as we can subdivide any polygon into triangles.

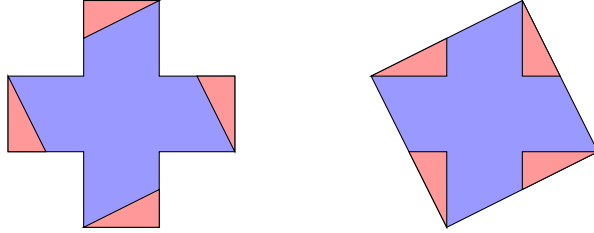


Figure 1. A cross and a square are equidecomposable.

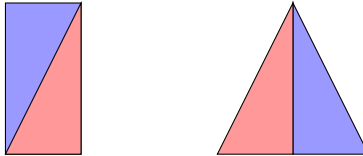


Figure 2. A rectangle is equidecomposable with *some* triangle of the same area.

There is an obvious criterion that needs to be satisfied in order for two polygons to be equidecomposable: they must have the same area. (If you're worried about Banach–Tarski-style issues here, don't be: we are insisting that they be cut up into polygons, not crazy nonmeasurable sets.)

But is area enough? Might it be the case that there are two polygons with the same area that are not equidecomposable? The Wallace–Bolyai–Gerwien Theorem [Ger33] says that this cannot be: area is the only invariant up to equidecomposability.

Theorem 2.5 (Wallace–Bolyai–Gerwien). *Two polygons P and Q are equidecomposable if and only if they have the same area.*

Proof. Let us perform some reductions to make the problem more tractable. We will show that if P is a polygon of area A , then P is equidecomposable with a rectangle of sidelengths 1 and A . Thus if P and Q both have area A , then they are both equidecomposable with this rectangle, and hence with each other, since we may freely reverse the cutting and rearranging operation.

In order to show that P is equidecomposable with a $1 \times A$ rectangle, we reduce to the case where P is a triangle. Certainly we may decompose any polygon into a bunch of triangles. Then, after cutting up each triangle and rearranging them into rectangles, we may place the rectangles side by side, yielding a rearrangement of the entire polygon.

So, it remains to show that a triangle of area A is equidecomposable with a $1 \times A$ rectangle. To do that, we will show that both of them are equidecomposable with a square of area A (hence of sidelength $\sqrt{A}$). And in fact, we don't need to do the rectangle separately, because we can easily rearrange the rectangle into *some* triangle of the same area; see Figure 2.

So, all that's left now is to show that any triangle is equidecomposable with a square of the same area. The first step is to rearrange the triangle into a parallelogram. We do this by making a cut halfway down the triangle parallel to one of the sides, flipping the small piece, and lining it up with the bottom piece, as shown in Figure 3.

The next step is to rearrange a parallelogram into a rectangle. This can also be easily done, as shown in Figure 4. This construction only works when the horizontal displacement between the two horizontal sides does not exceed the horizontal sidelength. When the

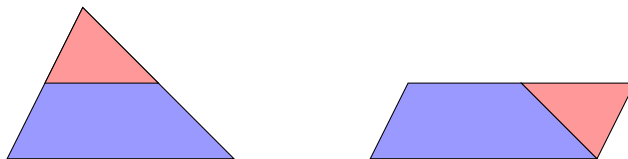


Figure 3. A triangle is equidecomposable with a parallelogram.

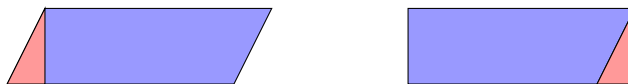


Figure 4. A parallelogram is equidecomposable with *some* rectangle.

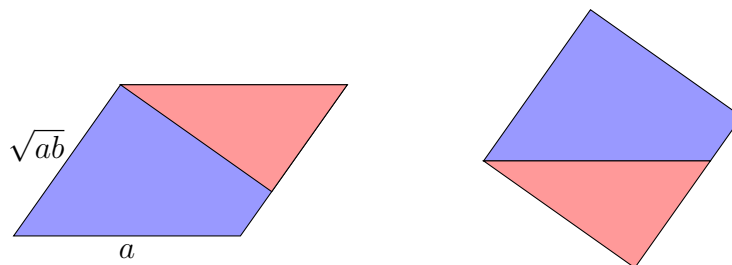


Figure 5. Rearranging an $a \times \sqrt{ab}$ parallelogram with area ab into a square of sidelength $\sqrt{ab}$.

horizontal displacement is larger, first cut the parallelogram horizontally into several congruent parallelograms, each of which has horizontal displacement smaller than the horizontal sidelength, split each one into a rectangle, and then stack the rectangles.

Now, even though we just rearranged a parallelogram into a rectangle, it's time to return to the parallelogram world, just with a more useful parallelogram. Suppose the rectangle has sides a and b , with $a > b$. Then we rearrange it into a parallelogram with sides a and $\sqrt{ab}$, using the reverse of the previous step. Note that this can be done, because the hypotenuse of the triangle cut out can have any side length between b and $\sqrt{a^2 + b^2}$, and certainly $\sqrt{ab}$ is somewhere in this range.

Finally, we take a parallelogram with area ab and sides a and $\sqrt{ab}$ and rearrange it into a square of sidelength $\sqrt{ab}$, as shown in Figure 5.

Putting all this together, we have first rearranged an arbitrary triangle into a square of the same area and then into a rectangle with one side equal to 1. If we do this for every triangle in an arbitrary triangulation of the original polygon, we may put the rectangles together into a large rectangle with one sidelength 1. Thus any polygon with area A is equidecomposable with a $1 \times A$ rectangle, and hence any two polygons with area A are equidecomposable. ■

3. POLYHEDRA AND HILBERT'S THIRD PROBLEM

So, now that we know that any two polygons with the same area are equidecomposable, we may move up a dimension and ask the same question about polyhedra. Note that we must now replace area with volume. This is Hilbert's Third Problem, at least in its modern formulation:

Question 3.1. *Given two polyhedra P and Q with the same volume, is it necessarily true that P and Q are equidecomposable?*

But here, the answer is different.

Theorem 3.2 (Dehn [Deh00]). *A cube and a regular tetrahedron of the same volume are not equidecomposable.*

In order to show that two polyhedra are *not* equidecomposable, the logical thing to do is to find an *invariant* that separates them. What this means is that we look for some function f on the set of polyhedra so that if P and Q are equidecomposable, then $f(P) = f(Q)$.

There are lots of such invariants. Some of them are silly. For example, we could define f so that $f(P) = 0$ for every polyhedron. While this satisfies our definition of an invariant, it isn't a very useful one. The reason is that one measure of the strength of an invariant is its ability to distinguish non-equidecomposable polyhedra. Our definition doesn't require that if P and Q are *not* equidecomposable, then $f(P) \neq f(Q)$, and that may be too difficult of a problem (although it would be fantastic if we could find such an invariant and prove it!). However, we would at least like it to be the case that *sometimes* when P and Q are not equidecomposable, then $f(P) \neq f(Q)$. The more often that holds, the better of an invariant it is.

Of course, we are already familiar with one invariant: the volume. Hilbert's Third Problem asks whether the volume is the *only* invariant, in the sense that if two polyhedra are not equidecomposable, then we can already detect that knowing solely their volumes. And Dehn proved that this is not true. He did this by finding a second invariant, known as the *Dehn invariant*.

Now, it's possible to define a second invariant very easily, but also in a silly way. Define $f(P) = 1$ if P is equidecomposable with a cube, and $f(P) = 0$ if P is not equidecomposable with a cube. While this invariant certainly distinguishes polyhedra that are equidecomposable with a cube from those that aren't, it does not give us much insight. In particular, we have no idea how to calculate it.

An invariant, as we defined it, is a rather flimsy thing. We didn't say, for example, where $f(P)$ should live. Frequently, but not always, it is a real number. We don't require invariants to have any particular properties other than being invariants, but often more structure is useful.

In the case of invariants for polyhedra, the following are useful:

- The invariant should be real-valued, i.e. for each polyhedron P , $f(P) \in \mathbb{R}$.
- The invariant should be *additive*, so that if we can cut P into sub-polyhedra $P_1, \dots, P_n$, then $f(P) = f(P_1) + f(P_2) + \dots + f(P_n)$. (A multiplicative invariant would be just as good, and we can convert between the two by taking exponentials or logarithms.)

Note that if a function on polyhedra is additive, then it is automatically an invariant, because if we can cut up P into pieces $P_1, \dots, P_n$ and rearrange the pieces into Q , then

$$f(P) = f(P_1) + \dots + f(P_n) = f(Q).$$

(Well, this is perhaps not quite right, since we need to ensure that f is a congruence invariant: if $P \cong Q$, then $f(P) = f(Q)$.)

So, now our goal is to construct an additive invariant that gives different answers for the cube and regular tetrahedron.

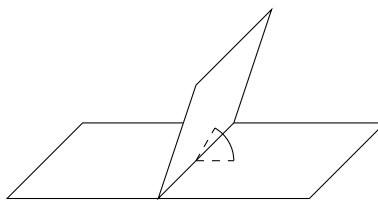


Figure 6. A dihedral angle.

4. THE DEHN INVARIANT

It is now time to define the Dehn invariant, which we will then use to give a negative answer to Hilbert's Third Problem. The Dehn invariant is defined in terms of the edge lengths and dihedral angles of a polyhedron.

Definition 4.1. Let e be an edge of a polyhedron; this means that two faces of the polyhedron meet at e . The *dihedral angle* at e is the angle between these two faces. We write α_e for the dihedral angle at e .

See Figure 6 for a picture of a dihedral angle.

Remark 4.2. In the rest of these notes, we always express our angles in radians.

The next thing we have to do is to define a very strange function: a function $g : \mathbb{R} \rightarrow \mathbb{R}$ with the following properties:

- $g(x + y) = g(x) + g(y)$ for all $x, y \in \mathbb{R}$. Such a function is said to be *additive*.
- $g(\pi) = 0$.

From these two conditions, we can deduce (see problem 7) that $g\left(\frac{m}{n}\pi\right) = 0$ for all rational numbers $\frac{m}{n}$. However, not all real numbers are rational multiples of π , so this does *not* imply that $g(x) = 0$ for all $x \in \mathbb{R}$, although this is one (unhelpful) possibility.

For an edge e in a polyhedron, let us write $|e|$ for its length. For a function g as above, we may now define the Dehn invariant with respect to g :

Definition 4.3. For an additive function g with $g(\pi) = 0$, we define the *Dehn invariant* of a polyhedron P with respect to g to be

$$D_g(P) = \sum_{e \in P} |e| \times g(\alpha_e).$$

The sum in the definition of the Dehn invariant is taken over all the edges of P .

It is not immediately obvious that the Dehn invariant is, in fact, an invariant. In order to prove that it is, we will show that it is additive: if we cut up P into little polyhedra $P_1, \dots, P_n$, then $D_g(P) = D_g(P_1) + \dots + D_g(P_n)$. As we discussed earlier, this is enough to guarantee that D_g is an equidecomposability invariant.

In order to show that D_g is an equidecomposability invariant, we need to investigate what happens when we take a polyhedron P and slice it in two by making a plane cut. Note that not all polyhedral subdivisions are obtainable by making a sequence of plane cuts, but if we have any polyhedral subdivision of P , then we can find a subdivision of that subdivision which is obtainable in this way: whenever we make a planar cut, we cut all the way along that plane, rather than potentially stopping partway through.

So, let's see what happens when we slice P by a plane cut L . There are two possibilities for how L intersects a face of P :

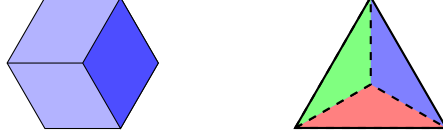


Figure 7. A cube and a regular tetrahedron of the same volume are not equidecomposable.

- (1) L intersects a face F of P in the middle of F , creating two new faces F_1 and F_2 . It does so by splitting two edges of F into two pieces, as well as creating a new edge which is common to both new faces. Let us look at how the Dehn invariant changes. Before the plane slice, the contribution to D_g coming from one of the edges e that was split into two pieces was $|e|g(\alpha_e)$. The plane slice doesn't change the dihedral angle, but it does change the length, by creating two pieces of length $|e_1|$ and $|e_2|$, where $|e_1| + |e_2| = |e|$. Thus the new contribution to the Dehn invariant is

$$|e_1|g(\alpha_e) + |e_2|g(\alpha_e) = (|e_1| + |e_2|)g(\alpha_e) = |e|g(\alpha_e).$$

This is the same as before, so the total Dehn invariant does not change with this type of intersection. Now let's look at the new edge e . Its dihedral angles with L are α'_e and α''_e . Note that $\alpha'_e + \alpha''_e = \pi$. Thus the contribution to the Dehn invariant of this new edge is

$$|e|g(\alpha'_e) + |e|g(\alpha''_e) = |e|g(\alpha'_e + \alpha''_e) = |e|g(\pi) = 0.$$

Thus this new edge contributes nothing to the Dehn invariant.

- (2) L intersects P along an edge of a face F , thus subdividing the dihedral angle. Let us suppose that the old dihedral angle at e was α_e , and the two new angles formed are α'_e and α''_e . Then the new contribution to the Dehn invariant is

$$|e|g(\alpha'_e) + |e|g(\alpha''_e) = |e|g(\alpha'_e + \alpha''_e) = |e|g(\alpha_e).$$

Again, the contribution at e to the Dehn invariant does not change.

Thus we see that, regardless of how L intersects P , the Dehn invariant does not change. Thus D_g is additive, and hence an equidecomposability invariant.

5. SOLVING HILBERT'S THIRD PROBLEM

With the Dehn invariant at hand, we may now solve Hilbert's Third Problem by proving Theorem 3.2: a cube and a regular tetrahedron are not equidecomposable; see Figure 7.

Proof of Theorem 3.2. We must compute the Dehn invariant of the cube and the tetrahedron. Let us scale both of them to have volume 1. In the case of the cube, this means that each edge has length 1. The dihedral angles are all $\frac{\pi}{2}$. The cube has 12 edges, and each one has a contribution of $|e|g(\alpha_e) = 1 \times g\left(\frac{\pi}{2}\right) = 0$ to the Dehn invariant. Thus the Dehn invariant is $D_g(\text{cube}) = 12 \times 0 = 0$. (Note that this does not depend on the choice of g . However, the Dehn invariant of the tetrahedron *will* depend on g .)

We must now compute the Dehn invariant of the tetrahedron. One can check (see problem 3) that, in order for the tetrahedron to have volume 1, the sidelength must be $\sqrt[3]{6\sqrt{2}}$. The dihedral angle is $\arccos\left(\frac{1}{3}\right)$; see problem 8. There are six edges, so the Dehn invariant of the tetrahedron is $6\sqrt[3]{6\sqrt{2}} \times g\left(\arccos\left(\frac{1}{3}\right)\right)$.

It's not completely obvious what this is. Is $\arccos\left(\frac{1}{3}\right)$ a rational multiple of π ? No!

Theorem 5.1. $\arccos\left(\frac{1}{3}\right)$ is not a rational multiple of π .

See [Niv56] for a book about proving irrationality.

Proof. Suppose we had

$$\arccos\left(\frac{1}{3}\right) = \frac{m}{n}\pi,$$

for some rational number $\frac{m}{n}$. Then, taking cosines of both sides, we would have

$$\cos\left(\frac{m}{n}\pi\right) = \frac{1}{3}.$$

We now show that this is impossible. (See problem 9 for a generalization.)

To show that $\alpha = \arccos\left(\frac{1}{3}\right)$ is not a rational multiple of π , we look at $\cos(k\alpha)$ for various integers k . In order to do this, we use the well-known trigonometric identity

$$\cos\theta + \cos\phi = 2\cos\frac{\theta+\phi}{2}\cos\frac{\theta-\phi}{2}$$

with $\theta = (k+1)\alpha$ and $\phi = (k-1)\alpha$ to get

$$\cos((k+1)\alpha) = 2\cos\alpha\cos k\alpha - \cos((k-1)\alpha)$$

for $k \geq 1$. By induction, we find that

$$\cos k\alpha = \frac{A_k}{3^k},$$

where A_k is an integer not divisible by 3; in fact,

$$A_{k+1} = 2A_k - 9A_{k-1}.$$

This shows that, if A_k is not divisible by 3, then A_{k+1} is also not divisible by 3.

Now, if α is a rational multiple of π , so $\alpha = \frac{m\pi}{n}$, then we can rewrite this as $n\alpha = m\pi$. Since $\cos(m\pi) = \pm 1$, we also have $\cos(n\alpha) = \pm 1$, so

$$\pm 1 = \cos(n\alpha) = \frac{A_n}{3^n}.$$

But this would imply that $A_n = \pm 3^n$. However, we showed earlier that A_n is not a multiple of 3, so this cannot happen. ■

We may now conclude the proof of Theorem 3.2. Since we have shown that $\arccos\left(\frac{1}{3}\right)$ is not a rational multiple of π , this means that there is *some* additive function g on $\mathbb{R}$ with $g(\pi) = 0$ so that $g\left(\arccos\left(\frac{1}{3}\right)\right) \neq 0$. (However, it is not possible to write down an actual function; this relies on the Axiom of Choice. It was first shown that such a function g exists in [Ham05].) Thus, for such a g , $D_g(\text{tetrahedron}) \neq 0$, so the cube and the tetrahedron are not equidecomposable. ■

6. MORE ON EQUIDECOMPOSABILITY

The Dehn invariant shows us how to conclude that two polyhedra are *not* equidecomposable. In order to show that two polyhedra *are* equidecomposable, we can find a way of cutting one and rearranging it into the other, but this relies on cleverness (or luck) every time. It would be nice to have a sort of converse to Dehn's result. Fortunately, there is such a result:

Theorem 6.1 (Sydler [Syd65]). *If P and Q are polyhedra of the same volume such that $D_g(P) = D_g(Q)$ for all g , then P and Q are equidecomposable.*

In other words, the Dehn invariants with respect to all g , together with volume, form a *complete* equidecomposability invariant: they completely classify polyhedra up to equidecomposability. This result is much harder to prove than Dehn's and is beyond the scope of these notes.

We may also be interested in higher dimensions. A *polytope* is a higher-dimensional analogue of a polyhedron. A typical example of a polytope in 4 dimensions is the 4-dimensional cube, sometimes also called the hypercube. We can represent it in $\mathbb{R}^4$ as $\{(x, y, z, w) \in \mathbb{R}^4 : 0 \leq x, y, z, w \leq 1\}$.

Given two polytopes P and Q of the same dimension, we can ask whether they are equidecomposable. Certainly they have to have the same n -dimensional volume. And there is an analogue of the Dehn invariant as well. It is defined just the same way as the three-dimensional Dehn invariant is, except with $(n - 2)$ -dimensional faces instead of edges:

$$D_g(P) = \sum_{f \in P} |f| g(\alpha_f),$$

where the sum is over the $(n - 2)$ -dimensional faces of P , $|f|$ denotes the $(n - 2)$ -dimensional *area* of the face, and α_f denotes the dihedral angle formed by the two $(n - 1)$ -dimensional facets meeting at that face. (In other words, everything moves up by $n - 3$ dimensions.) There is also an extension of Sydler's theorem to polytopes in $\mathbb{R}^4$: two polytopes are equidecomposable if and only if they have the same 4-dimensional volume and Dehn invariant for all functions g satisfying the conditions above; see [Jes72]. However, many questions remain open in higher dimensions, and it is likely that having the same volume and Dehn invariant is not sufficient for equidecomposability. See [Dup01] for a book on the subject.

7. BILINEAR FUNCTIONS AND THE TENSOR PRODUCT

There is a slight annoyance in the definition of the Dehn invariant: we have to define the Dehn invariant with respect to some additive function g . But the choice of function doesn't really matter all that much. It would be more aesthetically pleasing if we could remove the dependence on the choice of g . If we remove the g , then the Dehn invariant is a sum of products of terms of the form $|e| \alpha_e$, so that we are "multiplying" a length by an angle. But remember that $g(\pi) = 0$, so we need to make sure that this "product" is 0 whenever $\alpha_e = \pi$, or more generally whenever α_e is any rational multiple of π . Multiplication of real numbers does not have this property. Before we can define our new sort of multiplication, we need to abstract the notion of real numbers into more general *abelian groups*.

Definition 7.1. An *abelian group* is a set A together with a binary operation $+$, thought of as addition, satisfying the following properties:

- For any $a, b \in A$, $a + b \in A$.
- For any $a, b \in A$, $a + b = b + a$.
- For any $a, b, c \in A$, $a + (b + c) = (a + b) + c$.
- There is an element $0 \in A$, called the *identity element*, so that, for every $a \in A$, $a + 0 = a$.
- For every $a \in A$, there is a $b \in A$, called the *inverse* of a , so that $a + b = 0$. We usually write $-a$ for b .

The idea is that an abelian group is something that has the same fundamental *properties* as the real numbers under addition, but while not necessarily giving the same *answers*. Let's look at some examples.

Example. The integers under addition form an abelian group.

Example. Let m be a positive integer. Then the *integers modulo m* form an abelian group. There are many interpretations of this group, but one of them is to let the elements be $\{0, 1, \dots, m-1\}$ and define $a + b$ to be the *remainder* when $a + b$ is divided by m . For example, if $m = 10$, then $9 + 7 = 6$. The identity element 0 is still 0 , and $-a$ is $m - a$ for $a \neq 0$. Another, often more useful, way of thinking about this group is that its elements are all the integers, except that a and b are considered to be equal if they leave the same remainder upon division by m , i.e. if $a - b$ is a multiple of m . We denote this group by $\mathbb{Z}/m\mathbb{Z}$, and we write $n \pmod{m}$ for n considered as an element of $\mathbb{Z}/m\mathbb{Z}$.

Example. Similarly, consider the group $\mathbb{R}/\mathbb{Z}$ of *fractional parts* of real numbers. Its elements are the real numbers, except that we consider two real numbers a and b to be equal if they have the same fractional part, i.e. if $a - b \in \mathbb{Z}$.

Example. More generally, suppose that A is an abelian group and B is a *subgroup*, i.e. a subset which is also a group (with the same operation). Then we may define the group A/B to be A except that a_1 and a_2 are considered to be equal if $a_1 - a_2 \in B$. This is again a group; it is not too difficult to check this using the definition. (See problem 10.) The group A/B is called the *quotient group* of A by B and is pronounced “ A mod B .”

Notation. If A is an abelian group, $a \in A$, and $n \in \mathbb{Z}$, we define $n \cdot a$ or na to be $a + a + \dots + a$ (n times) if $n \geq 0$ and $-a - a - \dots - a$ ($-n$ times) if $n < 0$.

With this in mind, let us look back at how we intend to achieve our goal of reformulating the Dehn invariant. The edge lengths can be any positive real number. The positive real numbers don't form a group, but the most logical group they are part of is the group of real numbers. Some real numbers (the nonpositive ones) don't occur as edge lengths, but that's okay; we'll still be able to define our multiplication on them. Now, what about the angles? The angles are also real numbers, except that we require that rational multiples of π should “be” 0 . This is just what the quotient group is designed to do: it makes a certain subgroup (here the rational multiples of π) into 0 . Let us write $\pi\mathbb{Q}$ for the rational multiples of π . Then the group of angles in question is simply the quotient group $\mathbb{R}/\pi\mathbb{Q}$.

So, now we are focused on our goal: we need to learn how to multiply an element of $\mathbb{R}$ with an element of $\mathbb{R}/\pi\mathbb{Q}$. There is no particularly good way of doing this and getting a real number. (It can be done, but not in any systematic manner, and it requires the Axiom of Choice at its ugliest. This is not what we want.) A much better solution is to allow this product to live in some other abelian group. This group is the *tensor product* $\mathbb{R} \otimes (\mathbb{R}/\pi\mathbb{Q})$. Let us now build up to it.

In order to build up to the tensor product, we need to know what we want out of multiplication. As with addition, we want something with the same properties as multiplication. The key property here is the *distributive property*:

$$a \cdot (b_1 + b_2) = a \cdot b_1 + a \cdot b_2, \quad (a_1 + a_2) \cdot b = a_1 \cdot b + a_2 \cdot b.$$

We now mimic this property in order to define a sort of multiplication of two groups.

Definition 7.2. Let A, B, C be abelian groups. A *bilinear map* on $A \times B$ is a function $f : A \times B \rightarrow C$ satisfying the following properties:

- $f(a_1 + a_2, b) = f(a_1, b) + f(a_2, b)$ for all $a_1, a_2 \in A$ and $b \in B$
- $f(a, b_1 + b_2) = f(a, b_1) + f(a, b_2)$ for all $a \in A$ and $b_1, b_2 \in B$.

A bilinear map is a way of multiplying an element of A with an element of B and ending up with an element of C . So, we want to multiply a real number with an element of $\mathbb{R}/\pi\mathbb{Q}$, but what do we want C to be? Where do we want this product to live? One idea would be to choose C to be a familiar group like $\mathbb{R}$. Indeed, this is what we get when we choose an additive function g that vanishes on the rational multiples of π and define $f(r, \bar{\theta}) = r \cdot g(\theta)$, as we did earlier, where $r \in \mathbb{R}$, $\theta \in \mathbb{R}$, and $\bar{\theta}$ is the class of θ in $\mathbb{R}/\pi\mathbb{Q}$. But we would really like to do this *without* choosing a function g . Instead, we want to choose C to be the “best” possible group, specially built to handle multiplication of $\mathbb{R}$ with $\mathbb{R}/\pi\mathbb{Q}$. There are two ways of defining this group, known as the tensor product. One way is fairly concrete and involves representing it as a quotient of a (rather large) group. However, in practice, this method is not helpful at all; doing computations in this group is next to impossible, and from that definition, even determining whether something is zero is a herculean task.

The far better solution is to define it as a group with certain properties in terms of *other* groups. In order to do that, we need to discuss the relation between pairs of groups, and in particular functions between groups.

Definition 7.3. Let A and B be two abelian groups. A *homomorphism* from A to B is a function $f : A \rightarrow B$ so that $f(a_1 + a_2) = f(a_1) + f(a_2)$.

Thus a homomorphism is a function between groups that preserves the fundamental group structure, namely the addition.

Example. Let $A = \mathbb{Z}$ and $B = \mathbb{Z}/m\mathbb{Z}$ for some positive integer m . We define a homomorphism $f : A \rightarrow B$ by setting $f(n) = n \pmod{m}$ for each integer n .

Example. If A and B are any two abelian groups, we have at least one homomorphism from A to B , given by $f(a) = 0_B$ for all $a \in A$.

Example. If $B \subseteq A$ are two abelian groups, then we have a homomorphism $\iota : B \rightarrow A$ given by $\iota(b) = b$ for all $b \in B$, where $\iota(b)$ is considered as an element of A .

Example. Let B be any abelian group. Then we have several homomorphisms $f : \mathbb{Z} \rightarrow B$. Pick any $b \in B$, and define $f(n) = n \cdot b$ for all $n \in \mathbb{Z}$. This is a homomorphism.

Example. Let B be any abelian group. We will describe the homomorphisms from $\mathbb{Z}/m\mathbb{Z}$ to B . If $b \in B$ is an element such that $m \cdot b = 0$, then we can define a homomorphism $f : \mathbb{Z}/m\mathbb{Z} \rightarrow B$ by setting $f(n) = n \cdot b$. (In problem 11, you will determine what goes wrong if $m \cdot b \neq 0$.)

Now we’re *finally* ready to define the tensor product.

Definition 7.4. Let A and B be two abelian groups. Their tensor product $A \otimes B$ is an abelian group together with a bilinear map $f : A \times B \rightarrow A \otimes B$ with the following *universal property*: if C is any abelian group and $g : A \times B \rightarrow C$ is any bilinear map, then there is a unique homomorphism $\phi : A \otimes B \rightarrow C$ of abelian groups such that $g(a, b) = \phi(f(a, b))$ for all $a \in A$ and $b \in B$.

Remark 7.5. We frequently write $a \otimes b$ for $f(a, b)$.

Remark 7.6. We often express the relation $g = \phi \circ f$ by saying the diagram (or triangle)

$$\begin{array}{ccc} A \times B & \xrightarrow{f} & A \otimes B \\ & \searrow g & \downarrow \phi \\ & & C \end{array}$$

commutes.

This definition does not feel like a definition. After all, it seems to be challenging to verify that something is a tensor product of two groups. However, this can often be done.

Example. Let A be any abelian group. We will show that $A \otimes \mathbb{Z} = A$. First, we must define the bilinear map $f : A \times \mathbb{Z} \rightarrow A$. This map is $f(a, n) = na$. Now, suppose that $g : A \times \mathbb{Z} \rightarrow C$ is any bilinear map. We need to construct $\phi : A \rightarrow C$ so that $g(a, n) = \phi(an)$. (This is what the universal property says in this case.) Thus, to define $\phi(a)$, we just set $\phi(a) = g(a, 1)$. Note that we have the desired factorization: $\phi(f(a, n)) = \phi(na) = n \cdot g(a, 1) = g(a, n)$. Certainly ϕ is unique, because we don't have any other choices for how to define ϕ : $\phi(a)$ *must* be $g(a, 1)$ in order for the triangle to commute. But is it a homomorphism? Suppose $a, b \in A$. We must show that $\phi(a + b) = \phi(a) + \phi(b)$. Using the definition of ϕ , this means that we must show that $g(a + b, 1) = g(a, 1) + g(b, 1)$. But this is true, since we required that g be bilinear.

Returning to the Dehn invariant, recall that we want to define a multiplication of $\mathbb{R}$ with $\mathbb{R}/\pi\mathbb{Q}$. The best way of doing this is with the tensor product: $\mathbb{R} \otimes (\mathbb{R}/\pi\mathbb{Q})$. This group is not “obviously” equal (or *isomorphic*) to any familiar group. (It is non-obviously isomorphic to $\mathbb{R}$ under addition, but not in a useful way, and this is not really the right way of thinking about it.) The Dehn invariant is defined to be

$$D(P) = \sum_{e \in E(P)} |e| \otimes \bar{\alpha}_e \in \mathbb{R} \otimes (\mathbb{R}/\pi\mathbb{Q}),$$

where $E(P)$ is the set of edges of P and $\bar{\alpha}_e$ is the image of α_e in $\mathbb{R}/\pi\mathbb{Q}$. What Dehn's theorem shows is that if P and Q are two equidecomposable polyhedra, then they have the same Dehn invariant in $\mathbb{R} \otimes (\mathbb{R}/\pi\mathbb{Q})$. This is the best possible way of dealing with all D_g 's for all g 's simultaneously. For each g , we are defining a bilinear map from $\mathbb{R} \times (\mathbb{R}/\pi\mathbb{Q})$ to $\mathbb{R}$. By the definition of the tensor product, this means that it induces a homomorphism from $\mathbb{R} \otimes (\mathbb{R}/\pi\mathbb{Q})$ to $\mathbb{R}$. So, the tensor product packages up all these homomorphisms (or bilinear maps from $\mathbb{R} \times (\mathbb{R}/\pi\mathbb{Q})$), as well as all homomorphisms to other abelian groups, into one weird group. That is, for each suitable function g , we have a unique homomorphism $\Phi_g : \mathbb{R} \otimes (\mathbb{R}/\pi\mathbb{Q}) \rightarrow \mathbb{R}$ such that $\Phi_g(\ell \otimes \bar{\theta}) = \ell \cdot g(\theta)$. Then $D_g(P) = \Phi_g(D(P))$.

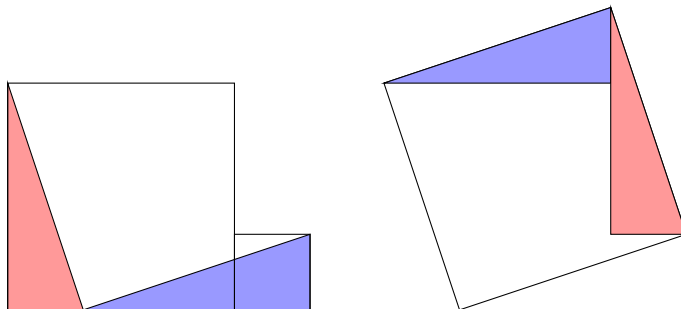


Figure 8. A proof of the Pythagorean Theorem.

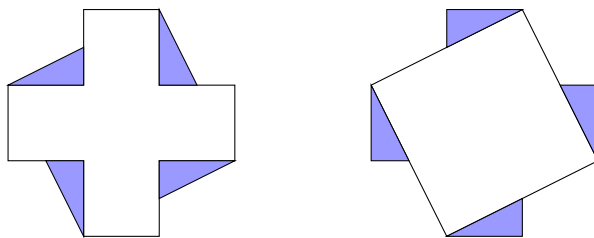


Figure 9. The cross and the square are equicomplementable.

8. PROBLEMS

- (1) Let H be a regular hexagon of sidelength 1, and let T be a right triangle with legs of length 3 and $\sqrt{3}$. (Note that H and T have the same area.) Explain how to dissect H into finitely many pieces and rearrange them into T .
- (2) Explain the proof of the Pythagorean Theorem alluded to in Figure 8.
- (3) Compute the volume of a regular tetrahedron with sidelength s .
- (4) Let T_1 be a regular tetrahedron, and let T_2 be a regular octahedron, both of the same volume. Are T_1 and T_2 equidecomposable? If so, how?
- (5) Suppose that T_1 and T_2 are two tetrahedra with the same base areas and the same heights. Are T_1 and T_2 necessarily equidecomposable?
- (6) We say that two polygons or polyhedra P and Q are *equicomplementable* if there are finitely many pairs of polygons or polyhedra $R_1 \cong R'_1, \dots, R_n \cong R'_n$ so that $P \cup R_1 \cup \dots \cup R_n$ and $Q \cup R'_1 \cup \dots \cup R'_n$ are partitions of congruent polygons or polyhedra. (See Figure 9.) Show that two polygons are equidecomposable if and only if they are equicomplementable.
- (7) Suppose that $g : \mathbb{R} \rightarrow \mathbb{R}$ is an additive function such that $g(\pi) = 0$. Show that $g\left(\frac{m}{n}\pi\right) = 0$ for all $\frac{m}{n} \in \mathbb{Q}$.
- (8) Show that the dihedral angles of a regular tetrahedron are $\arccos\left(\frac{1}{3}\right)$.
- (9) Show that if n is an integer greater than or equal to 3, then $\arccos\left(\frac{1}{n}\right)$ is not a rational multiple of π .
- (10) Show that if A is an abelian group and B is a subgroup of A , then A/B is an abelian group.
- (11) When describing the homomorphisms from $\mathbb{Z}/m\mathbb{Z}$ to an abelian group B , we picked an element $b \in B$ with $m \cdot b = 0$ and defined $f(n) = n \cdot b$. What goes wrong if $m \cdot b \neq 0$?
- (12) Show that $A \otimes B = B \otimes A$.

- (13) Let m be a positive integer. Show that, for any abelian group A , $A \otimes (\mathbb{Z}/m\mathbb{Z}) = A/mA$, where mA is the subgroup of A consisting of all elements of the form ma for $a \in A$. Conclude that $(\mathbb{Z}/m\mathbb{Z}) \otimes (\mathbb{Z}/n\mathbb{Z}) = \mathbb{Z}/g\mathbb{Z}$, where $g = \gcd(m, n)$.
- (14) Show that $\mathbb{Z}^m \otimes \mathbb{Z}^n = \mathbb{Z}^{mn}$.
- (15) We call an element of $A \otimes B$ of the form $a \otimes b$ an *elementary tensor*. It is tempting to think that a typical element of $A \otimes B$ can be written as an elementary tensor. However, this is not true in general: a typical element of $A \otimes B$ is merely a *finite sum* of elements of this form. Find an element of $\mathbb{Z}^2 \otimes \mathbb{Z}^3$ which is not an elementary tensor.
- (16) The tensor product is far from the only construction to have a universal property. Here are a few others:
- (a) Show that the quotient group A/B satisfies the following universal property: let B be a subgroup of A and let $\pi : A \rightarrow A/B$ be the map from A to A/B . If C is any abelian group and $f : A \rightarrow C$ is a homomorphism so that $f(b) = 0_C$ for all $b \in B$, then there is a unique homomorphism $g : A/B \rightarrow C$ such that $f(a) = g(\pi(a))$ for all $a \in A$.
 - (b) Let A and B be two abelian groups. Define their *direct product* $A \times B$ to be the group consisting of elements (a, b) where $a \in A$ and $b \in B$, with sum $(a_1, b_1) + (a_2, b_2) = (a_1 + a_2, b_1 + b_2)$. There are homomorphisms $\pi_1 : A \times B \rightarrow A$ and $\pi_2 : A \times B \rightarrow B$ defined by $\pi_1(a, b) = a$ and $\pi_2(a, b) = b$. Show that, for any abelian group C with homomorphisms $\phi_1 : C \rightarrow A$ and $\phi_2 : C \rightarrow B$, there is a unique homomorphism $\psi : C \rightarrow A \times B$ so that $\phi_i(c) = \pi_i(\psi(c))$ for $i = 1, 2$.
- (17) (a) It is well-known that the only regular convex polyhedra are the tetrahedron, cube, octahedron, dodecahedron, and icosahedron. But can you prove it?
- (b) Classify the convex regular polytopes in $\mathbb{R}^4$. What about higher dimensions? See [Cox73]

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KALEIDOSCOPE CIRCLES, MOUNTAIN VIEW, CA 94040

E-mail address: simon@kalcircles.com