HOW CLOSE CAN A CHROMATIC ROOT GET TO ONE?

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ABSTRACT. In this paper we show that the chromatic function $P(G, t)$ is always a polynomial. We then ask a simple question: where does this polynomial equal zero? Tutte showed that two stretches of the real line, $(-\infty, 0)$ and $(0, 1)$, can never contain a root. Jackson showed in 1993 that no root can lie between 1 and 32/27 either, and that this bound is sharp. We prove both of these results from scratch, along with all the tools needed to get there, and we close by looking at what has been discovered about this threshold since 1993.

1. Introduction

In 1912, George Birkhoff, an American mathematician, introduced a polynomial invariant of a graph with one specific goal in mind: to try and solve the Four Color Problem, which had been open since the 1850s and asked a question that looked very simple at first glance: Whether every map drawn in a plane could have its regions colored with only four colors so that no two adjacent regions share a color? His main idea was to count the number of valid colorings as a function of the number of available colors t, and then to try to show that this function is positive at $t = 4$ for every planar map. Two decades later, Hassler Whitney proved that this function is always a polynomial in t, for every graph, and developed much of the theory and ideas that are used up to this date and throughout this paper. The Four Color Problem itself was finally solved in 1976 by Appel and Haken by a computational method that did not use the chromatic polynomial at all. But the polynomial survived as another important subfield.

In this paper, we study this chromatic polynomial as an algebraic polynomial, but we preserve its ideas related to graph theory and introduce some of them as well. We then establish that $P(G, t)$ is a polynomial. Then we establish that we can plug in any real number t by treating the chromatic polynomial algebraically. The values of t where $P(G, t) = 0$ are called chromatic roots, similar to the roots of an algebraic polynomial, and we then study their bounds and distribution on the number line, which turns out to be very surprising and involves the use of a very arbitrary but particular constant, 32/27.

In the 1970s, William Tutte showed that the intervals $(-\infty, 0)$ and $(0, 1)$ contain no chromatic root of any graph whatsoever. This means every nontrivial chromatic root is at least 1. This raised the natural question of how close to 1, from above, a chromatic root can get, and whether there is a specific bound to how close it can get. The answer, proved by Bill Jackson in 1993, is also the central result of this paper: no graph has a chromatic root that lies between 1 and $\frac{32}{27}$, and this bound cannot be improved, since there is an explicit family of graphs whose roots approach $\frac{32}{27}$ arbitrarily closely. Carsten Thomassen later completed the picture by showing that above, or $> \frac{32}{27}$, chromatic roots are everywhere: they are dense in the $(\frac{32}{27}, \infty)$ region and do not have any other gaps, as seen in the cases $(-\infty, 0)$, $(0, 1)$, and $(1, \frac{32}{27})$.

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This paper is a self-contained introduction to these ideas of chromatic polynomials and their interpretation as algebraic polynomials. We assume the reader is familiar with the basic definition of a graph but nothing else. Section 2 defines proper colorings and the chromatic number. Section 3 introduces the chromatic polynomial and proves it is always a genuine polynomial. Section 4 develops deletion-contraction, the main computational tool, and works out explicit formulas for several families of graphs. Section 5 defines chromatic roots and proves Tutte's theorem in full. Section 6 introduces generalized triangles, states Jackson's theorem, and gives a complete proof. Section 7 surveys results since 1993.

This exposition tries to summarize this complex set of ideas into a single paper to make it easy for beginners and readers to understand the complex ideas of chromatic polynomials and their bounds when they are treated as algebraic polynomials.

2. Graphs and colorings

Mathematics is the art of giving the same name to different things - Henri Poincaré This art is seen in graph theory almost all the time. Think of it as modeling real life situations, the same graph can represent different paths in google maps, a network of friendships, or the connections between neurons in our brain. If we strip away every real life meaning attached to the dots and lines, what we are left with, in each case, is the exact same underlying graph. In this section we define the basics of graphs and graph colorings that will be used throughout this paper, and we try to keep the definitions as simple as possible.

Definition 2.1. A graph $G = (V, E)$ consists of a set V of vertices (also called nodes) and a set E of edges.

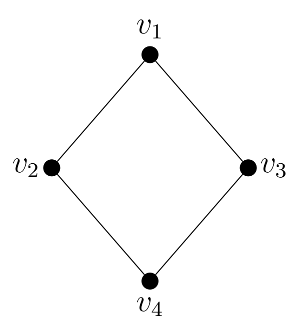

Figure 1. A graph with four vertices and four edges.

Definition 2.2. Two vertices that are joined by an edge are called adjacent vertices.

Definition 2.3. Let G be a graph and let t be a positive integer. A proper t-coloring of G assigns each vertex one of t colors, written as a function

$$c\colon V(G) \to \{1, \ldots, t\}$$

such that $c(u) \neq c(v)$ whenever u and v are adjacent.

Definition 2.4. The chromatic number of a graph G, written $\chi(G)$, is the smallest positive integer t for which a proper t-coloring of G exists.

The definition 2.4 above might be too abstract thus to make it click for the readers we will now perform two examples.

2 colors fail: b, c forced to clash

3 colors succeed

Figure 2. $\chi(K_3) = 3$.

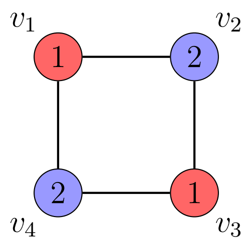

Figure 3. 2 colors succeed: $\chi(C_4) = 2$.

Example. In K_3 every vertex is adjacent to every other, so all three colors must differ. In C_4, opposite vertices are not adjacent and can share a color. The above examples show the idea of graph colorings, in which we assign a color to a node such that adjacent nodes do not have the same color. Moreover, we then determine the minimum number of colors required to satisfy the condition that adjacent nodes do not have the same color. We then find that, in the case of the K_3 graph, the chromatic number is 3, and in the case of the C_4 graph, the chromatic number is 2.

Definition 2.5. The complete graph K_n is the graph on n vertices in which every pair of distinct vertices is adjacent.

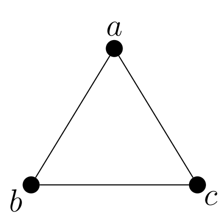

Figure 4. The complete graph K_3, the triangle.

Definition 2.6. The cycle C_n, for $n \geq 3$, is the graph on vertices $v_1, v_2, \ldots, v_n$ in which v_i is adjacent to v_{i+1} for each i (indices read modulo n, so v_n is adjacent to v_1), and there are no other edges.

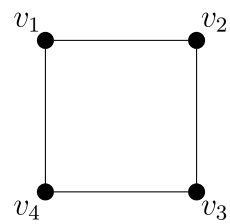

Figure 5. The cycle C_4.

Definition 2.7. The complete bipartite graph $K_{m,n}$ consists of two disjoint sets of vertices, one of size m and one of size n, in which every vertex in the first set is adjacent to every vertex in the second set, and there are no edges within either set.

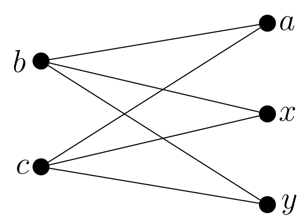

Figure 6. The complete bipartite graph $K_{2,3}$, with parts $\{b, c\}$ and $\{a, x, y\}$.

Definition 2.8. A tree is a connected graph that contains no cycles. A vertex of degree exactly 1 in a tree is called a leaf.

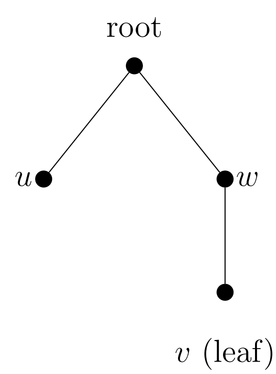

Figure 7. A tree. The vertex v has degree 1 and is a leaf.

Definition 2.9. An edge e of a graph G is called a bridge if its removal increases the number of connected components of G.

Definition 2.10. The connected components of a graph G are the maximal subsets of $V(G)$ within which every two vertices are joined by a path using edges of G. We write $c(G)$ for the number of connected components of G.

3. The Chromatic Polynomial

In the last section we formalised and defined the basic definitions in graph theory and graph colorings. The fundamental idea of Definition 2.4 was the chromatic number, a number which determines the minimum number of t colors that can be used to color the nodes of a graph G such that no two adjacent nodes have the same color. We now move on to a similar, yet somehow different, idea: for any number of colors t, how many different valid colorings are there? To answer this question, we start by understanding a chromatic function and then by proving that chromatic function of a simple graph is a polynomial.

Definition 3.1. Let G be a graph and let t be a positive integer. We write $P(G, t)$ for the number of distinct proper t-colorings of G, as defined in Definition 2.3. We call $P(G, t)$, a function of t, the chromatic function of G.

Theorem 3.2. *The chromatic function of a simple graph is a polynomial.*

Proof. Let G be a graph on n vertices, and recall that an independent set is a set of vertices, no two of which are adjacent. Just to make it clear again, an independent set can be thought of as a way arranging and dissecting a graph into sets such that each set consists of vertices or nodes that are not joined by an edge and each node in an independent set has the same color as any other node in that independent set. For each integer k with $0 \leq k \leq n$, let e_k denote the number of ways to partition $V(G)$ into exactly k nonempty independent sets. Every proper t-coloring of G uses some number k of distinct colors, and the vertices receiving each particular color form an independent set (since adjacent vertices are never colored the same). So a proper t-coloring using exactly k colors corresponds to a choice of a partition of $V(G)$ into k nonempty independent sets, together with an assignment of k distinct colors, taken from the t available, to those k parts. There are e_k choices of partition, and for each one, $t(t-1)(t-2)\cdots(t-k+1)$ ways to assign distinct colors to its k parts. Summing over all possible values of k, every proper coloring is counted exactly once, so

$$P(G, t) = \sum_{k=0}^{n} e_k \cdot t(t-1)(t-2)\cdots(t-k+1).$$

Each term in this sum is a polynomial in t (the term for $k = 0$ being the constant 0, when G is nonempty), and a finite sum of polynomials is a polynomial. Hence $P(G, t)$ is a polynomial. ■

Corollary 3.3. *The chromatic polynomial of G on n vertices has degree exactly n with leading coefficient* 1.

Proof. The term for index k in the sum has degree k, so the highest degree term is the $k = n$ term, which has degree n. All other terms have degree strictly less than n, so the overall degree is n. The coefficient of t^n comes entirely from the $k = n$ term, $e_n \cdot t(t-1)\cdots(t-n+1)$. Since $e_n = 1$ (the only partition of n vertices into n nonempty independent sets places each vertex in its own singleton), the leading coefficient is 1. ■

Remark 3.4. We have now proved that $P(G, t)$ is a polynomial in t. Throughout the rest of the paper we will refer to it as the chromatic polynomial of G.

Remark 3.5. We now substitute different values of t in $P(G, t) = t(t-1)(t-2)$ for the graph K_3. We start with $t = 1$ which gives us $P(G, t) = 1(1-1)(1-2)$ this equates to 0. We then put $t = 2$ which gives us $P(G, t) = 2(2-1)(2-2)$ which equates to 0. We then put

$t = 3$ which gives us $P(G, t) = 3(3-1)(3-2)$ which equates to 6. We notice that the first nonzero value occurs at $t = 3$ which is the same as the chromatic number of K_3 that we defined earlier.

Proposition 3.6. *$P(G, 0) = 0$ for any graph with at least one vertex.*

Proof. With 0 colors available no vertex can receive a color so the count is 0. ■

Proposition 3.7. *$P(G, 1) = 0$ whenever G has at least one edge.*

Proof. With only 1 color every vertex receives that color, so any edge connects two vertices with matching colors, violating the rule and forming a non valid coloring. ■

Proposition 3.8. *If G has n vertices and no edges then $P(G, t) = t^n$.*

Proof. With no edges there are no constraints, each vertex independently chooses from t colors, and n independent choices multiply to t^n. ■

Proposition 3.9. *If G is the disjoint union of G_1 and G_2 then $P(G, t) = P(G_1, t)P(G_2, t)$.*

Proof. Coloring G_1 and G_2 are entirely independent since no edges connect them, and independent choices multiply. ■

While the formula in Theorem 3.2 proves $P(G, t)$ is always a polynomial, computing e_k for a complicated graph is very difficult in practice as it requires enumerating all possible partitions into independent sets. Therefore, to make this simpler we introduce a new tool, the deletion-contraction method, which computes $P(G, t)$ for any graph by breaking it into two simpler graphs. We use it to derive closed formulas for trees, cycles, and $K_{2,3}$. This method simplifies this process and helps in actually computing $P(G, t)$ comparatively easily.

4. Deletion-Contraction and Examples

We now define a very elegant theorem with its proof, the Deletion-Contraction theorem. To give you an overview before we start, this theorem is basically a recursive way of breaking down graphs into simpler graphs. Now, you may ask why is this helpful? Well, in the K_3 example we calculate the chromatic polynomial for a simple graph but this may or may not be that easy for other graphs. We might end up encountering much more complex graphs and arranging them into independent sets and then counting is a lot harder. Therefore, to break down these graphs at each stage and doing it recursively to obtain the most simple form of these graphs so that you can calculate the final chromatic polynomial is one of the most elegant ideas in this paper. We will start by proving the Deletion-Contraction theorem and then use it to derive formulas for trees, complete graphs and $K_{2,3}$.

Definition 4.1. The graph $G - e$ is obtained from G by removing the edge e while keeping both of the vertices that defined e in the graph G.

Definition 4.2. The graph G/e is obtained from G by removing the edge e and merging both of the vertices that defined e in the graph G into a single vertex.

Before looking at the proof I would like to give some intuition behind the whole idea. So the main way that I like to visualize is that you have a graph G that has two nodes and one edge connecting them. Now what you do is that you make two copies of this graph G, G_1 and G_2. But in G_1 you remove the edge connecting the two nodes and in G_2 you remove

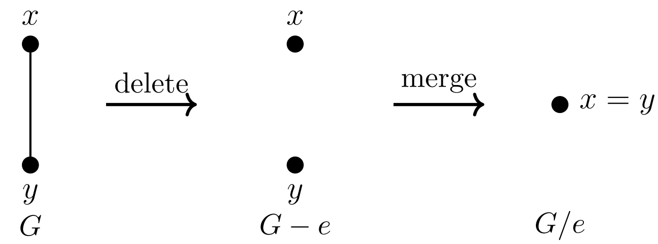

Figure 8. The edge $e = xy$ in G, the graph $G - e$ after deletion, the graph G/e after contraction.

the edge and merge the two nodes to one node. We now assume that every graph in this case follows a proper coloring or has a proper t-coloring. We now start with graph G, since G has two nodes joined by an edge both the nodes need to have different colors, we call this the different coloring case. For graph G_1 both the nodes are not joined by an edge so they can have any color they want, we call this the any coloring possible case. For graph G_2 both the nodes are merged to one node so this one node can only have one color, we call this the same coloring case. Now if you think of it the different coloring case and the same coloring case add up to form the any coloring possible case. Mathematically, we give this as $P(G-e, t) = P(G, t) + P(G/e, t)$. We now move on to the actual proof.

Theorem 4.3 (Deletion-Contraction)**.** *For any graph G and any edge e of G,*

$$P(G, t) = P(G - e,\, t) - P(G/e,\, t).$$

Proof. Consider all proper t-colorings of $G - e$. Since the edge e between x and y is absent in $G - e$, there is no rule forcing x and y to receive different colors. Every proper t-coloring of $G - e$ therefore falls into exactly one of the following two cases.

Case 1: x and y receive different colors. The only constraint that G poses over the constraints of $G - e$ is that x and y must differ. Since they already do in this case, every such coloring of $G - e$ is automatically a valid proper coloring of G, and conversely every proper coloring of G gives a coloring of $G-e$ where x and y differ. So the number of colorings of $G - e$ in this case is exactly $P(G, t)$.

Case 2: x and y receive the same color. Since x and y always carry the same color in this case, they behave as a single merged vertex with respect to any other vertex in the graph. This is precisely the situation in the contracted graph G/e, where x and y have been merged. So the number of colorings of $G - e$ in this case is exactly $P(G/e, t)$.

Since every proper coloring of $G - e$ falls into exactly one of the two cases,

$$P(G - e,\, t) = P(G, t) + P(G/e,\, t).$$

Rearranging the equation above gives us $P(G, t) = P(G - e,\, t) - P(G/e,\, t)$. ■

Remark 4.4. Theorem 4.3 gives a second proof that $P(G, t)$ is always a polynomial. We induct on the number of edges of G. If G has no edges then $P(G, t) = t^n$, which is a polynomial. For the inductive step, pick any edge e. By induction $P(G - e, t)$ and $P(G/e, t)$ are both polynomials (each graph has fewer edges than G), and a difference of two polynomials is a polynomial. Hence $P(G, t)$ is a polynomial.

Example. Complete graph K_n. We show that

$$P(K_n,\, t) = t(t-1)(t-2)\cdots(t-n+1).$$

Order the vertices $v_1, v_2, \ldots, v_n$ arbitrarily. Color them one at a time. Vertex v_1 may receive any of the t available colors. Vertex v_2 is adjacent to v_1 and must therefore differ from it, leaving $t-1$ choices. Vertex v_3 is adjacent to both v_1 and v_2, which already carry two distinct colors, so $t-2$ choices remain. Continuing in this way, vertex v_k is adjacent to all of $v_1, \ldots, v_{k-1}$, which together carry $k-1$ distinct colors, leaving $t-k+1$ choices. Multiplying the independent choices at each step,

$$P(K_n, t) = t(t-1)(t-2)\cdots(t-n+1).$$

In particular, for $n = 3$ this gives $P(K_3, t) = t(t-1)(t-2)$, which is consistent with our computation in Section 3 using the sum formula.

Example. Tree on n vertices. We show that for any tree T on n vertices,

$$P(T,\, t) = t(t-1)^{n-1}.$$

We prove this by induction on n. For the base case $n = 1$, the tree is a single vertex with no edges, so $P(T, t) = t = t(t-1)^0$, which matches the formula.

For the inductive step, let $n \geq 2$ and suppose the formula holds for all trees on fewer than n vertices. Every tree on at least two vertices has a leaf: a vertex v of degree exactly 1. Let u be the unique neighbour of v. Remove v and its single edge from T; the resulting graph $T' = T - v$ is a tree on $n-1$ vertices (it remains connected and gains no cycle by this removal), so by the inductive hypothesis $P(T', t) = t(t-1)^{n-2}$.

To extend any proper colouring of T' to all of T, we must assign a colour to v. Since v is adjacent only to u, the only constraint on v is that it must differ from u's colour. This leaves exactly $t-1$ choices for v, regardless of which colour u received. Therefore

$$P(T, t) = (t-1) \cdot P(T', t) = (t-1) \cdot t(t-1)^{n-2} = t(t-1)^{n-1}.$$

Example. A Complete bipartite graph $K_{2,3}$. Label the two vertices on the left b, c and the three vertices on the right a, x, y, so that every vertex in $\{b, c\}$ is adjacent to every vertex in $\{a, x, y\}$ and there are no edges within either group. We show that

$$P(K_{2,3},\, t) = t(t-1)^3 + t(t-1)(t-2)^3.$$

The cleanest way to count colorings here is to think about what happens to b and c first, since every vertex on the right side depends on both of them.

Case 1: b and c get the same color. There are t ways to choose this shared color. Each of a, x, y now just needs to avoid this one color, leaving $t-1$ choices for each of them independently. This case contributes $t(t-1)^3$.

Case 2: b and c get different colors. There are $t(t-1)$ ways to assign two distinct colors to b and c. Now each of a, x, y must avoid both of these colors, leaving $t-2$ choices for each of them independently. This case contributes $t(t-1)(t-2)^3$.

Since every coloring falls into exactly one of these two cases, we add:

$$P(K_{2,3}, t) = t(t-1)^3 + t(t-1)(t-2)^3 = t(t-1)\left[(t-1)^2 + (t-2)^3\right] = t(t-1)(t^3 - 5t^2 + 10t - 7).$$

The cubic factor $t^3 - 5t^2 + 10t - 7$ is going to come back in the next section in a very important way, as the source of the first nontrivial chromatic root above 1.

With these examples we now have explicit chromatic polynomials for the main families of graphs we will need. But there is something interesting already happening: we have been treating t as a positive integer that counts the number of available colors. But $P(G, t)$ is a polynomial, and polynomials do not care what you plug into them. If we treat this chromatic polynomial algebraically then you can plug in any real number t, including fractions and negative numbers, and the formula still produces a number. This naturally raises the question: for which real values of t does $P(G,t) = 0$? We cover this in the upcoming sections and this question forms the main thesis of this paper.

5. Chromatic Roots and Tutte's Theorem

In the last few sections we defined Chromatic numbers and Chromatic polynomials. We now have an idea of what $P(G,t)$ exactly is and we have also proved that it is always a polynomial. We now strip away all the graphs and ask an algebraic question about the chromatic polynomial $P(G,t)$: Can we have any value of t that is real that can be substituted in the polynomial? As it turns out, the answer is yes. From a purely algebraic point of view, a chromatic polynomial is like any other polynomial in algebra and we can plug in any real value of t into it.

Since it is a polynomial it must also have a root: a value of t that when substituted in the polynomial gives 0 as an output. We then ask, for what values of t does $P(G,t) = 0$? These values do not have to be positive integers, and the answer turns out to be a lot more structured and bounded than one might expect.

Definition 5.1. A *chromatic root* of a graph G is a value of t at which $P(G,t) = 0$.

We already know two chromatic roots that every graph with at least one edge automatically has, directly from Propositions 3.6 and 3.7: $t = 0$ and $t = 1$. These are not at all important to us.

Definition 5.2. The chromatic roots $t = 0$ and $t = 1$ are called the trivial roots of G. Any other chromatic root is called nontrivial.

So the interesting question here is that very do these non trivial roots live? Do they have a specific bound that can make finding them a lot more easier? It turns out there is.

The answer to the previous question was given by Mathematician William Tutte. He came up with a proof that showed that no graph's chromatic polynomial has a root in $(-\infty, 0)$ or in $(0, 1)$.

Theorem 5.3 (Tutte)**.** *No graph has a chromatic root in* $(-\infty, 0)$ *or in* $(0, 1)$.

The proof of this theorem is given by a very original and creative idea which will also be used partially or fully in the other proofs that we will establish further in this paper.

The proof tracks the sign of a renormalized version of $P(G,t)$ through deletion-contraction induction. The main idea is that if a quantity is always strictly positive somewhere, it can never be zero there, so no root can exist in that region.

Proof. We split the proof into two parts, one for each interval: $(-\infty, 0)$ and $(0, 1)$.

Part A: no roots below 0.

We claim that for every graph G on n vertices, $(-1)^n P(G,t) > 0$ for all real $t < 0$.

We prove this by induction on the number of edges of G. If G has no edges then $P(G,t) = t^n$ by Proposition 3.8, so $(-1)^n P(G,t) = (-1)^n t^n = (-t)^n$. Since $t < 0$ we have $-t > 0$, and any positive number raised to a positive integer power is positive. So the base case holds.

For the inductive step, suppose G has at least one edge e. Both $G-e$ (same n vertices, fewer edges) and G/e ($n-1$ vertices, fewer edges) have strictly fewer edges than G, so the inductive hypothesis applies to both. By induction, $(-1)^n P(G-e,t) > 0$ and $(-1)^{n-1} P(G/e,t) > 0$. Since $(-1)^{n-1} = -(-1)^n$, the second inequality says $(-1)^n P(G/e,t) < 0$, which means $-(-1)^n P(G/e,t) > 0$.

By Theorem 4.3, $P(G,t) = P(G-e,t) - P(G/e,t)$, so

$$(-1)^n P(G,t) = (-1)^n P(G-e,t) + \big(-(-1)^n P(G/e,t)\big).$$

This is a sum of two positive quantities, hence positive. The induction is complete, and since $(-1)^n P(G,t) > 0$ for all $t < 0$, we have $P(G,t) \neq 0$ there.

Part B: no roots in $(0,1)$.

This part is more delicate. We need to also keep track of the number of connected components $c(G)$ of the graph. We claim that for every graph G on n vertices with $c = c(G)$ connected components, $(-1)^{n+c} P(G,t) > 0$ for all real $t \in (0,1)$.

Again we induct on the number of edges. If G has no edges, then every vertex is isolated, so $c = n$ and $P(G,t) = t^n$. Then $(-1)^{n+c} P(G,t) = (-1)^{2n} t^n = t^n > 0$ since $t > 0$. Base case holds.

For the inductive step, we pick any edge e of G and split into two cases depending on whether e is a bridge.

Case 1: e is not a bridge. Deleting a non-bridge does not change the number of connected components: $c(G-e) = c(G) = c$. Contracting never changes the component count either, so $c(G/e) = c$, while G/e has $n-1$ vertices. By the same sign argument as Part A, with $n + c$ playing the role of n, both inductive hypotheses give positive quantities and the sum is positive.

Case 2: e is a bridge. When e is a bridge, removing it splits the component containing e into two pieces. Say the graph decomposes as two subgraphs H_1 and H_2 (the pieces on either side of the bridge), with n_1, c_1 and n_2, c_2 vertices and components respectively, so that $n = n_1 + n_2$ and $c = c_1 + c_2 - 1$ (the bridge was connecting one component from each side into one).

For this case we use the following identity. When G consists of two pieces H_1 and H_2 joined by exactly one edge from vertex $x \in H_1$ to vertex $y \in H_2$,

$$P(G,t) = P(H_1,t) \cdot P(H_2,t) \cdot \frac{t-1}{t}.$$

To see why: the total number of unrestricted proper colorings of H_1 and H_2 independently is $P(H_1,t) \cdot P(H_2,t)$. By color symmetry, for any fixed coloring of H_1, exactly $\frac{1}{t}$ of the colorings of H_2 will assign vertex y the same color as x. Subtracting those invalid pairs,

$$P(G,t) = P(H_1,t) \cdot P(H_2,t) - P(H_1,t) \cdot \frac{P(H_2,t)}{t} = P(H_1,t) \cdot P(H_2,t) \cdot \frac{t-1}{t}.$$

Now we track the sign. We have $(-1)^{n+c} = (-1)^{n_1+n_2+c_1+c_2-1} = -(-1)^{n_1+c_1}(-1)^{n_2+c_2}$. By induction on H_1 and H_2 (both have fewer edges than G), $(-1)^{n_1+c_1} P(H_1,t) > 0$ and $(-1)^{n_2+c_2} P(H_2,t) > 0$. So

$$(-1)^{n+c} P(G,t) = -(-1)^{n_1+c_1} P(H_1,t) \cdot (-1)^{n_2+c_2} P(H_2,t) \cdot \frac{t-1}{t}.$$

For $t \in (0,1)$: $t > 0$ so $\frac{1}{t} > 0$, but $t - 1 < 0$ so $\frac{t-1}{t} < 0$. The expression above is therefore $-(\text{positive})(\text{positive})(\text{negative}) = -(\text{negative}) > 0$.

The induction is complete, and since $(-1)^{n+c} P(G,t) > 0$ for all $t \in (0,1)$, we have $P(G,t) \neq 0$ there. ■

Tutte's theorem tells us something very important: the entire real line below 0 and the entire open interval $(0, 1)$ do not have any non trivial chromatic root. Every nontrivial chromatic root must therefore satisfy $t \geq 1$.

This raises the question we encountered earlier in a sharper form: we know roots must be at least 1, and we saw that $K_{2,3}$ has a root at about 1.431. Can we find graphs with roots closer to 1? Say at 1.2? Or 1.1? Or arbitrarily close to 1? The answer turns out to be no, and the precise boundary of what is possible is the remarkable number $\frac{32}{27}$.

6. Generalized Triangles and Jackson's Theorem

We now arrive at the heart of this paper, perhaps the most important theorem of this whole paper. We will construct an explicit infinite family of graphs whose smallest nontrivial chromatic roots decrease monotonically toward a specific limiting value, and then state the theorem that this value is an absolute bound: no graph of any kind can have a nontrivial chromatic root below it.

The construction starts with a single operation that we define as a double subdivision.

Definition 6.1. Let G be a graph and let uv be an edge of G. The double subdivision of G at uv is the graph obtained by deleting the edge uv, adding two new vertices x and y, and adding the four edges ux, xv, uy, yv. In other words, the single direct connection between u and v is replaced by two separate two-step paths through the new vertices.

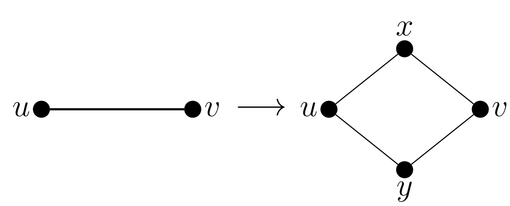

Figure 9. Double subdivision of edge uv: the single edge becomes two parallel paths of length two through new vertices x and y.

Definition 6.2. A generalized triangle is either K_3 itself, or any graph obtained from K_3 by a finite sequence of double subdivisions, each applied to some edge present at that stage. We write $\mathcal{K}$ for the family of all generalized triangles.

Generalized triangles matter because they turn out to be the extremal family for this entire problem: as we will see, they are simultaneously the objects that Jackson's proof reduces every graph to, and the objects whose roots get closest to the bound. Understanding this one family is therefore enough to understand the whole phenomenon.

The first natural question that arises is: what does one double subdivision produce? Take K_3 with vertices a, b, c and apply one double subdivision to the edge bc: delete bc, add new vertices x and y, and add edges bx, xc, by, yc. The resulting graph has vertices $\{a, b, c, x, y\}$. Looking at the adjacencies: b is adjacent to a, x, y and c is adjacent to a, x, y. There are no edges within $\{b, c\}$ and no edges within $\{a, x, y\}$. This is precisely the bipartite structure of $K_{2,3}$ with parts $\{b, c\}$ and $\{a, x, y\}$.

So $K_{2,3}$ is the first generalized triangle after K_3 itself, and we showed in Example 4 that its smallest nontrivial root is approximately 1.431.

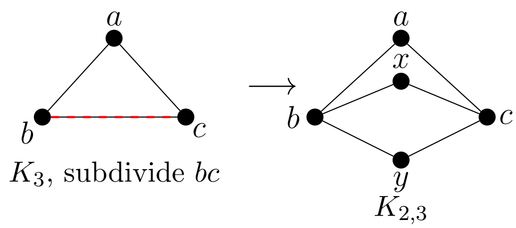

Figure 10. One double subdivision of K_3 at edge bc (dashed) produces exactly $K_{2,3}$ with parts $\{b, c\}$ and $\{a, x, y\}$.

We now build one specific and explicit sequence of generalized triangles. It is important to be clear about what this sequence is for: it is not the family that Jackson's proof inducts over, and it is not how the theorem is actually established. Rather, it is the extremal family that shows the bound $\frac{32}{27}$ cannot be improved, by showing roots that get arbitrarily close to it. We introduce it now so that we can present a table of shrinking roots that will better help understand this, and we will return to its precise role once we have discussed the structure of the proof itself.

Definition 6.3. Set $G_1 = K_{2,3}$, obtained from K_3 by double-subdividing the edge bc as above. Let p, q denote the vertices b, c: these two vertices exist through every later step, since at each stage we subdivide one of the two length-two paths between them rather than an edge incident to p or q directly. For $k \geq 1$, having built G_k, we let G_{k+1} be the graph obtained from G_k by double-subdividing one of the two paths of length two between p and q. This gives an explicit sequence $G_1, G_2, G_3, \ldots$ with $|V(G_k)| = 2k + 3$.

Continuing this construction produces the following pattern of roots as represented by the table below.

Graph	$\lvert V\rvert$	Smallest nontrivial real root
K_3	3	none in $(1, 2)$
$G_1 = K_{2,3}$	5	≈ 1.4302
G_2	7	≈ 1.3611
G_3	9	≈ 1.3351
$\vdots$		$\downarrow$
		$\to \frac{32}{27} \approx 1.1852$

Each successive graph in the sequence has a smaller root, and these roots decrease monotonically toward $\frac{32}{27}$, getting arbitrarily close but never reaching it. This sequence is what makes the bound in Jackson's theorem: it shows that no number larger than $\frac{32}{27}$ could replace it.

Theorem 6.4 (Jackson)**.** *No graph has a chromatic root in the open interval $\left(1, \frac{32}{27}\right)$. This bound is sharp: the sequence $G_1, G_2, G_3, \ldots$ of Definition 6.3 has smallest nontrivial real roots converging to $\frac{32}{27}$ from above, so $\frac{32}{27}$ cannot be replaced by any larger number.*

We will now describe the structure of Jackson's proof without reproducing it in full, since the complete argument requires a very lengthy analysis that goes well beyond the scope of this exposition. We want to be really precise about what the proof actually does, since it is very easy to describe it in a wrong way because of the nuances it has. We also clarify some basic mistakes that people made while understanding this proof.

The proof is not simply an induction along the sequence $G_1, G_2, \ldots$. That sequence is only the extremal example that shows sharpness. Jackson's theorem is a statement about every graph, and the proof has to handle every graph, not just the members of one recursively built family using the theorems we established before.

The proof has two separate parts, and the harder one comes first. The first and by far the longer part of Jackson's argument is a structural reduction: assuming, for contradiction, that some graph has a chromatic root in $(1, \frac{32}{27})$, one takes a graph G with the fewest vertices among all such counterexamples. Jackson then shows, through a detailed case analysis of the possible structure of a minimal counterexample, that G must in fact be a generalized triangle. This is a counterexample argument in the same spirit as many extremal graph theory proofs: rather than directly showing that every graph does not lie in a specific interval, one shows that if a counterexample exists at all, the smallest one must belong to the much more restricted and structured family $\mathcal{K}$. Establishing this reduction, and ruling out every other possible structure a minimal counterexample could have, occupies most of Jackson's paper. It is the main idea of the theorem.

Only after this reduction is complete does the second, shorter part of the proof begin: verifying, by an inductive sign argument, that no generalized triangle actually has a root in $(1, \frac{32}{27})$. This second part is the part that is similar to the sign-tracking induction we used for Tutte's theorem in Section 5, and it is the part we can say something very surely about in this paper.

The mathematical tool needed for the second part is a general two-vertex separation formula, not the simple bridge formula. Every generalized triangle beyond K_3 has a 2-vertex cut: a pair of vertices $\{u, v\}$ whose removal splits the graph into two pieces. This is a structural fact about how double subdivisions build up a graph, though we do not attempt a general proof of it here beyond the specific sequence G_k, where it can be checked directly: removing the two vertices p, q from G_k separates the most recently added one from the rest of the graph. For an arbitrary generalized triangle, the two pieces on either side of such a cut need not take any single simple shape (a 4-cycle, in particular, is not the general picture; that is specific to the sequence G_k, where one side of the cut at the final step happens to be a 4-cycle). The main difficulty is that the two separator vertices u, v are typically not adjacent in G (the edge between them was deleted at some earlier subdivision step), so the bridge identity from Tutte's proof, which required an actual edge joining the two pieces, does not apply. Jackson's argument instead uses a general two-vertex separation formula that splits colorings of the whole graph according to whether u and v receive the same color or different colors, and expresses $P(G, t)$ in terms of the chromatic polynomials of the two pieces together with the edge uv added.

The threshold $\frac{32}{27}$ is seen in this analysis as a sharp inequality, not as a value derived from any single formula. As we work through the sign-tracking argument with this general separation formula, Jackson's analysis produces an inequality that holds throughout $(1, \frac{32}{27})$ and fails after it. We state this as the outcome of Jackson's computation rather than re-deriving the inequality here.

Remark 6.5. Jackson's analysis shows that two phenomena coincide at the same number: the threshold below which his sign-tracking inequality is guaranteed to hold, and the limiting value of the smallest root along the extremal sequence G_k. That these two independently defined quantities turn out to be the same number, $\frac{32}{27}$, is what makes the bound really special and established.

The sharpness of this Theorem 6.4 also follows from the table above: since the roots of G_k converge to $\frac{32}{27}$ from above, no larger constant could replace $\frac{32}{27}$ in the statement of the theorem. Future expository treatments, including work of Dong and Koh and of Perrett, revisit this reduction to generalized triangles and place it in the broader context of minor-closed families of graphs, which we discuss further in Section 7.

Thomassen completed this picture in 1997 by showing that above $\frac{32}{27}$, the situation is as dense as it possibly could be.

Theorem 6.6 (Thomassen)**.** *Chromatic roots are dense in* $\left(\frac{32}{27}, \infty\right)$*: for every real* $t > \frac{32}{27}$*, there exists a graph whose chromatic polynomial vanishes at* t*.*

Combining Jackson's and Thomassen's theorems give us a complete and really clean picture and idea: two forbidden zones from Tutte's theorem, a third forbidden zone from Jackson's theorem (which we established through the reduction to generalized triangles described above), and then complete density from $\frac{32}{27}$ onward with no further gaps. They establish and posit that the number $\frac{32}{27}$ is the unique number dividing line between the two areas of no non trivial roots and a dense network of them.

7. Results post Jackson's theorem since 1993

Jackson's theorem from 1993 answered the important question of where do the chromatic roots live on the real line, at least in terms of the big picture. But it consequently opened up a new set of questions: if you restrict to some specific family of graphs, does the so called forbidden zone get larger? It turns out that yes, it does, and several exact thresholds have been computed in the years since.

The main idea that makes this tractable is due to Dong and Koh, who identified a beautiful shortcut for these minor closed families of graphs. A family $\mathcal{G}$ of graphs is called minor-closed if whenever $G \in \mathcal{G}$, every graph obtainable from G by deleting edges, deleting vertices, or contracting edges also belongs to $\mathcal{G}$. In simple words, a minor-closed family is one where every simplified version of a member is still a member.

Theorem 7.1 (Dong and Koh)**.** *For any minor-closed family* $\mathcal{G}$*, the infimum of nontrivial chromatic roots over* $\mathcal{G}$ *equals the infimum over just the generalized triangles in* $\mathcal{G}$*.*

This is why generalized triangles have been the main objects that have been used throughout this paper: they are always the worst case examples, no matter which minor closed family you are looking at. If you want to find the lowest possible nontrivial chromatic root in any minor closed family, you only ever need to check the generalized triangles inside that family. Everything else will have a higher smallest root.

Using this principle, Dong and Koh were able to compute exact thresholds for several natural families defined by forbidding a small graph or a set of it as a minor. The results they got were as follows.

Forbidden minor	Threshold
K_3, $K_4{-}e$, or $K_{2,3}$	2
K_4	$\frac{32}{27}$
$K_{2,4}$	≈ 1.431

The third row is very important because a lot of people misread it. The threshold for graphs avoiding $K_{2,4}$ as a minor is approximately 1.431, which is also the root of $K_{2,3}$ that we computed in Example 4. The thing to notice here is that this is not the root of $K_{2,4}$: in fact, $K_{2,4}$ has no nontrivial real chromatic root at all. You can verify this yourself by computing $P(K_{2,4}, t) = t(t-1)^4 + t(t-1)(t-2)^4$ (using the same case analysis as for $K_{2,3}$) and checking that the remaining factor after extracting $t(t-1)$ has no real roots. The threshold 1.431 comes from $K_{2,3}$: it is the generalized triangle that sits just inside the $K_{2,4}$ minor free family and produces the lowest root. This is a nice illustration of the Dong-Koh principle in action.

Separate from the minor-closed story, Thomassen proved a threshold for a different kind of constraint/restriction.

Theorem 7.2 (Thomassen)**.** *Every graph containing a Hamiltonian path (a path that visits every vertex exactly once) has no chromatic root below* $t_0 \approx 1.296$*, where* t_0 *is the unique real root in* $(1, 2)$ *of the polynomial* $t^3 - 2t^2 + 4t - 4$.

Since $t_0 \approx 1.296$ is larger than $\frac{32}{27} \approx 1.185$, requiring a Hamiltonian path pushes the threshold up, which makes sense: most large generalized triangles do not have a Hamiltonian path, so requiring one rules out the graphs that produce roots closest to $\frac{32}{27}$.

Perrett extended this further in 2018, he did this by identifying specific families of generalized triangles with exact thresholds. One family has threshold exactly $\frac{5}{4} = 1.25$. Another has threshold equal to the unique real root in $(1, 2)$ of $t^4 - 4t^3 + 4t^2 - 4t + 4$, approximately 1.225. A third family, when restricted in the right way, gives a threshold coinciding exactly with Thomassen's $t_0 \approx 1.296$, and Perrett showed this coincidence is not an accident: the two families produce the same set of chromatic polynomials, so they have the same threshold. All of these results followed a similar pattern which led to the following conjecture, which is the open problem of the area.

Conjecture 7.3 (Perrett)**.** *Let* $\mathcal{G}$ *be a minor-closed family of graphs. Then the infimum of nontrivial chromatic roots over* $\mathcal{G}$ *is strictly greater than* $\frac{32}{27}$ *if and only if* $\mathcal{G}$ *does not contain every generalized triangle.*

In simple words, $\frac{32}{27}$ is special because it belongs to the full, unrestricted family of all graphs. Any restriction at all, as long as it rules out even one generalized triangle, should push the threshold strictly higher. This conjecture has been verified for all the specific families computed so far, but a proof in full generality remains open.

8. Conclusion

In this paper, we provide an exposition on the chromatic polynomials and their roots. We are able to establish through a set of theorems by Tutte, Jackson and Thomassen that no chromatic polynomial has a non trivial root between $(-\infty, 0)$, $(0, 1)$ and $(1, 32/27)$. We then conclude the paper with a short overview of recent work in this field and then define Perrett's conjecture which is a very big open problem right now.

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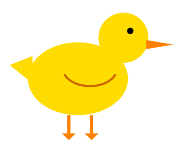
References

(1) G. D. Birkhoff, A determinant formula for the number of ways of coloring a map, Annals of Mathematics (Second Series), 14(1/4), 42–46.
(2) H. Whitney, The coloring of graphs, Annals of Mathematics (Second Series), 33(4), 688–718.
(3) W. T. Tutte, Chromials, in: Lecture Notes in Mathematics , Springer, 1974
(4) Jackson, B. (1993). A zero-free interval for chromatic polynomials of graphs. Combinatorics, Probability and Computing, 2(3), 325–336.
(5) C. Thomassen The zero-free intervals for chromatic polynomials of graphs. Combinatorics, Probability and Computing, 6(4), 497–506.
(6) C. Thomassen, Chromatic roots and Hamiltonian paths, Journal of Combinatorial Theory Series B, 80(2), 218–224.
(7) F. M. Dong and K. M. Koh, On zero-free intervals in $(1, 2)$ of chromatic polynomials of some families of graphs, SIAM Journal on Discrete Mathematics 24(2), 370–378.
(8) T. Perrett, Chromatic roots and minor-closed families of graphs, SIAM Journal on Discrete Mathematics, 31(1) (2018), 606–618.

Appendix: Manim Animation

An animation visualizing the chromatic polynomial and its roots, made in Manim, is available at the following link.

`https://youtu.be/2DqqaHNYBu0`



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