EULER CIRCLE IRPW: THE OUTER AUTOMORPHISM OF S_6

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Abstract. The symmetric groups are fundamental objects in abstract algebra, and for nearly every value of n, their automorphism groups are completely inner. The unique exception is the symmetric group S_6, which possesses a nontrivial outer automorphism. This paper develops the theory leading to this exception by introducing symmetric groups, automorphisms, and complete groups. After establishing why S_n is complete for all $n \neq 2, 6$ and later proving it, we investigate the exceptional structure of S_6. We present several explicit constructions of its outer automorphism, including the six mystic pentagons, labeled triangles and labeled icosahedra as well as an intuitive interpretation arising from projective geometry. Each construction realizes the same automorphism by exhibiting a non-standard action of S_6 on a different six-element combinatorial or geometric object, showing that transpositions are mapped to triple transpositions and therefore cannot arise from conjugation. This unexpected symmetry is one of the most remarkable phenomena in finite group theory.

1. Introduction

Generally, one may pursue mathematics for two main reasons. First, one may enjoy the power and generality of the statements that can be proved in mathematics. Second, mathematics contains striking examples that exhibit unusual irregularities or unexpected beauty. This paper is motivated by the latter reason and explores the group S_6, which is a remarkable exception among the symmetric groups.

Abstract algebra is a relatively new branch of mathematics that emerged in the nineteenth century, largely due to the work of Évariste Galois [R]. The symmetric group S_n, consisting of all permutations of n elements, is found in many areas of abstract algebra. One of the central questions in group theory is to understand the symmetries of a group itself, known as its automorphisms. For most symmetric groups, every automorphism arises from relabeling the underlying elements, making all automorphisms inner. Consequently, S_n is a complete group for all $n \neq 2, 6$ [C].

The existence of an outer automorphism of S_6 is one of the most surprising results in finite group theory. Discovered in the nineteenth century through work related to projective geometry and later studied through combinatorial constructions, this structure has no analogue in any other symmetric group. Unlike inner automorphisms, which preserve cycle structure, the outer automorphism of S_6 exchanges transpositions with triple transpositions, revealing a hidden symmetry within the group.

The unusual nature of the outer automorphism of S_6 has made it a subject of interest since the nineteenth century. The first examples of this phenomenon arose from the work of Arthur Cayley and later Jordan, who studied permutation groups through geometric and combinatorial methods. The exceptional symmetry of six objects was first connected to projective geometry, where configurations of six points in the plane revealed hidden relationships between different ways of describing the same geometric structure. Later, mathematicians

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such as W. W. Rouse Ball, J. H. Conway [Co, RB], and others developed more combinatorial descriptions of the automorphism through objects such as synthemes, pentads, and the mystic pentagons. These constructions transformed the abstract existence of the outer automorphism into a visual and concrete phenomenon, demonstrating that the exceptional behavior of S_6 is not an isolated algebraic accident but reflects a deeper connection between algebra, geometry, and combinatorics.

Several elegant constructions of this automorphism have been developed, including those involving synthemes and pentads. We will primarily focus on a combinatorial interpretation called the six mystic pentagons [BHRD]. These graphical objects provide a concrete way to visualize the special symmetry of S_6. In this paper, we explore these constructions and prove that they lead to the unique outer automorphism of S_6.

2. An Introduction to Group Theory

Definition 2.1. A *group* can be defined as a set G that has an operation

$$\cdot : G \times G \to G$$

such that $\cdot$ satisfies:

- Closure: $\forall\ a, b \in G, a \cdot b \in G$.
- Identity: $\exists\ e \in G$ such that $\forall\ g \in G, e \cdot g = g \cdot e = g$.
- Inverses: $\forall\ g \in G\ \exists\ g^{-1} \in G$ such that $g \cdot g^{-1} = e$.
- Associativity: $\forall a, b, c \in G, (a \cdot b) \cdot c = a \cdot (b \cdot c)$.

This definition is quite abstract so we'll look at an example.

Example. The integers under addition are a group: the sum of any two integers is an integer which shows closure. We have that $\forall\ x \in \mathbb{Z},\ x + 0 = x$ which shows that 0 is the identity. We also know that $x + x^{-1} = 0$ for $x^{-1} = -x$ which shows that every x has an inverse, $-x$. Finally, we notice that grouping and order of elements does not matter which shows associativity. This is probably the simplest examples of a group, but in this paper we will look at symmetric groups which have a more interesting structure.

Definition 2.2. We denote the set of integers from 1 to n as $[n] = \{1, 2, ..., n\}$. The *symmetric group* S_n is defined as the set of bijections from $[n]$ to $[n]$. This is simply the permutations of the set $[n]$ which means that S_n has the order $n!$.

The best way to visualize the elements of S_n is using permutation cycle notation. For example,

$$S_3 = \{(1)(2)(3),\ (1\,2),\ (1\,3),\ (2\,3),\ (1\,2\,3),\ (1\,3\,2)\}.$$

Example. We will look at the permutation $(1\,2) \in S_3$ and its action on the set

$$[3] = \{1, 2, 3\}.$$

The cycle notation gives that $(1\,2)$ sends

$$1 \mapsto 2, \qquad 2 \mapsto 1, \qquad 3 \mapsto 3.$$

Hence,

$$(1\,2) \cdot \{1, 2, 3\} = \{\,(1\,2)(1),\ (1\,2)(2),\ (1\,2)(3)\,\} = \{2, 1, 3\} = \{1, 2, 3\}.$$

Example. Consider the permutations $(1\,3)$ and $(2\,3)$ in S_3. We compute

$$(1\,3)(2\,3).$$

Recall that permutations are composed from right to left, so we first apply $(2\,3)$ and then $(1\,3)$.

We can determine the image of each element separately:

$$(2\,3) \text{ sends } 1 \mapsto 1 \text{ and } (1\,3) \text{ sends } 1 \mapsto 3, \text{ so } 1 \mapsto 3$$

$$(2\,3) \text{ sends } 2 \mapsto 3 \text{ and } (1\,3) \text{ sends } 3 \mapsto 1, \text{ so } 2 \mapsto 1$$

$$(2\,3) \text{ sends } 3 \mapsto 2 \text{ and } (1\,3) \text{ sends } 2 \mapsto 2, \text{ so } 3 \mapsto 2$$

Thus

$$1 \mapsto 3, \qquad 2 \mapsto 1, \qquad 3 \mapsto 2,$$

which is the cycle

$$(1\,3\,2).$$

Hence,

$$(1\,3)(2\,3) = (1\,3\,2).$$

Definition 2.3. Let G and H be groups. A *group homomorphism* is a function $\phi : G \to H$ such that

$$\phi(xy) = \phi(x)\phi(y)$$

for all $x, y \in G$.

To understand the structure of a group, one of the most important things to look at is its automorphisms.

Definition 2.4. An *automorphism* of a group G is a map $\phi : G \to G$ that is bijective and homomorphic. We denote the automorphism group of G as $\mathrm{Aut}(G)$

Automorphisms must preserve the structure of a group and thus we notice some interesting properties.

Proposition 2.5. *Let $e \in G$ be the identity element. Then, an automorphism $\phi : G \to G$ must send the identity to itself, or $\phi(e) = e$.*

Proof. Since e is the identity, $e \cdot e = e$. Then if we apply ϕ to both sides we obtain

$$\phi(e \cdot e) = \phi(e)\phi(e) = \phi(e)$$

because ϕ is a homomorphism. Multiplying both sides by $\phi^{-1}(e)$ gives

$$\phi^{-1}(e)\phi(e) \cdot \phi(e) = \phi^{-1}(e)\phi(e),$$

so $\phi(e) = e$. Therefore, an automorphism must send the identity element to itself. ∎

Definition 2.6. Let G be a group and let N be a normal subgroup of G. The *quotient group* G/N is the set of left cosets of N in G:

$$G/N = \{gN : g \in G\}.$$

The group operation on G/N is defined by

$$(gN)(hN) = (gh)N.$$

Since N is normal, this operation is well-defined, and G/N forms a group called the *quotient group* of G by N.

There are two kinds of automorphisms:

An inner automorphism is an automorphism which can be expressed through conjugation of a fixed element of G. An automorphism ϕ is then inner if for some $g \in G$ then

$$\phi_g : G \to G$$

$$\phi_g(h) := g^{-1}hg$$

We denote the inner automorphism group of G as $\mathrm{Inn}(G)$.

An outer automorphism is simply an automorphism that is not inner. We denote the outer automorphism group by $\mathrm{Out}(G)$.

Proposition 2.7. $\mathrm{Out}(G) = \mathrm{Aut}(G)/\mathrm{Inn}(G)$

Proof. The main fact we need to prove here is that $Inn(G)$ is a normal subgroup of $Aut(G)$, because $\mathrm{Out}(G) = \mathrm{Aut}(G)/\mathrm{Inn}(G)$ is simply the definition of an outer automorphism:
Let $\phi \in \mathrm{Aut}(G)$ and $\pi_g \in \mathrm{Inn}(G)$. Then we consider the conjugate $\phi\pi_g\phi^{-1}$.
Notice that for any $x \in G$,

$$\phi\pi_g\phi^{-1}(x) = \phi(g\phi^{-1}(x)g^{-1}).$$

By the definition of a homomorphism, we see that

$$\phi(g\phi^{-1}(x)g^{-1}) = \phi(g) \cdot \phi(\phi^{-1}(x)) \cdot \phi(g^{-1}) = \phi(g)x\phi(g)^{-1}$$

We notice that this is conjugation by the element $\phi(g)$ so

$$(\phi\pi_g\phi^{-1})(x) = \pi_{\phi(g)}(x)$$

and thus $\phi\pi_g\phi^{-1} \in \mathrm{Inn}(G)$

■

We can also see more clearly why this definition $Out(G) = Aut(G)/Inn(G)$ works. The inner automorphism group contains automorphisms which differ by an inner automorphism (by conjugation) which means that all automorphisms which don't differ by an inner automorphism are outer automorphisms and belong to the quotient group.

Definition 2.8. We say that two elements $h, g \in G$ commute if and only if $h^{-1}gh = g$, which implies that $hg = gh$. Then we define the center, $Z(G)$ to be the set of all elements in a group that commute with all the other elements in the group:

$$Z(G) = \{g \in G \mid gx = xg \text{ for all } x \in G\}.$$

We call the center of G trivial if $Z(G) = e$ where e is the identity element. In the case of symmetric groups, the center is trivial $\forall\ S_n, n \neq 2$ which we will prove shortly. On the other hand, abelian groups are groups which have maximal centers, so $Z(G) = G$. The most simple example of this is once again the integers under addition.

Definition 2.9. Let $\phi : G \to H$ be a group homomorphism. The *kernel* of ϕ is the set

$$\ker(\phi) = \{g \in G \mid \phi(g) = e_H\},$$

where e_H is the identity element of H.

Generally, $Out(G)$ is more interesting then $Inn(G)$ because outer automorphisms arise from some unexpected group structure while inner automorphisms are quite predictable. This is what makes the outer automorphism of S_6 special, because there is no trivial way to find this automorphism.

3. Completeness of Symmetric Groups

Definition 3.1. We say a group G is complete if the center of G is trivial and $Aut(G) = Inn(G)$.

Theorem 3.2. *S_n is complete $\forall\ n \neq 2, 6$.*

A very important property within symmetric groups is that conjugation preserves the order of an element. This means that an inner automorphism must send an element of one cycle type to an element of the same cycle type.

Lemma 3.3. *Let $\sigma \in S_n$ and let $\tau \in S_n$. Then $\sigma\tau\sigma^{-1}$ has the same cycle type as τ. In particular, conjugation preserves cycle structure.*

Proof. It suffices to show that conjugation sends cycles to cycles of the same length.
Let $\tau = (a_1\, a_2\, \ldots\, a_k)$ be a cycle of length k in S_n, then $\tau(a_i) = a_{i+1}$. Consider the conjugate

$$\sigma\tau\sigma^{-1}.$$

Then we will compute its action of the elements of the form $\sigma(a_i)$.
For $1 \leq i \leq k$, we have

$$\sigma\tau\sigma^{-1}(\sigma(a_i)) = \sigma\tau(a_i) = \sigma(a_{i+1}).$$

Since we have a cycle take the indices modulo k, so $a_{k+1} = a_1$.
Thus $\sigma\tau\sigma^{-1}$ sends

$$\sigma(a_1) \mapsto \sigma(a_2) \mapsto \cdots \mapsto \sigma(a_k) \mapsto \sigma(a_1),$$

so

$$\sigma\tau\sigma^{-1} = (\sigma(a_1)\, \sigma(a_2)\, \ldots\, \sigma(a_k)),$$

which is again a cycle of length k.
Since any permutation decomposes uniquely into disjoint cycles, we see that conjugation preserves the lengths of each disjoint cycle and hence preserves cycle type. ■

In the case of S_6, we will show that there exists an automorphism that does not preserve cycle type, and thus the automorphism cannot be inner.

As an example we will prove that S_3 is complete.

Proposition 3.4. *The group S_3 is complete.*

Lemma 3.5. *For any $\tau \in S_n$ and any transposition $(i\ j)$,*

$$\tau(i\ j)\tau^{-1} = (\tau(i)\ \tau(j)).$$

We will not prove this lemma here as it follows from pure algebraic computation. We will also see many other similar properties of transpositions in section 5 when proving completeness for $S_n,\ n \neq 2, 6$ in general.

Proof. Let $\sigma \in Z(S_3)$. Since σ commutes with every element, in particular $\sigma(i\ j)\sigma^{-1} = (i\ j)$ for all $i \neq j$ in $\{1, 2, 3\}$. Combined with **Lemma 3.5**, this gives

$$\{\sigma(i), \sigma(j)\} = \{i, j\} \quad \text{for every pair } i \neq j.$$

Applying this to each of the three pairs from $\{1, 2, 3\}$ we see that

$$\{\sigma(1), \sigma(2)\} = \{1, 2\}, \qquad \{\sigma(1), \sigma(3)\} = \{1, 3\}, \qquad \{\sigma(2), \sigma(3)\} = \{2, 3\}.$$

Suppose $\sigma(1) = 2$. Then the first equation forces $\sigma(2) = 1$. But then the second equation says $\{2, \sigma(3)\} = \{1, 3\}$, which is impossible since $2 \notin \{1, 3\}$. So $\sigma(1) \neq 2$. Similarly, swapping the roles of 2 and 3, $\sigma(1) \neq 3$. Hence $\sigma(1) = 1$. The identical argument applied to the pairs $\{2, 3\}$ shows $\sigma(2) = 2$ and $\sigma(3) = 3$.
Thus since σ sends every element to itself $\sigma = e$, $Z(S_3) = \{e\}$.
Automorphisms preserve the order of elements. In S_3, direct enumeration shows that order 2 occurs only among the transpositions $(1\,2), (1\,3), (2\,3)$ (the identity has order 1 and the two 3-cycles have order 3). Hence every $\phi \in \mathrm{Aut}(S_3)$, preserving order, must send transpositions to transpositions. This means that ϕ permutes the set

$$T = \{(1\,2), (1\,3), (2\,3)\}.$$

This step is specific to S_3, as in general, preserving order does not force preserving cycle type. The outer automorphism of S_6 shows transpositions and products of three disjoint transpositions both have order 2 but lie in different conjugacy classes.
Considering T on the set $\{1, 2, 3\}$ gives a homomorphism

$$\Phi : \mathrm{Aut}(S_3) \mapsto S_3.$$

We claim that Φ is injective. If $\phi \in \ker(\Phi)$, then ϕ fixes each transposition. Since the transpositions generate S_3, it follows that ϕ is the identity automorphism. Hence Φ is injective, and so

$$|\mathrm{Aut}(S_3)| = |S_3| = 6.$$

Now both $\mathrm{Aut}(S_3)$ and S_3 have the same finite order at most 6, so Φ is an isomorphism, thus

$$\mathrm{Aut}(S_3) \cong S_3.$$

Since

$$G/\ker(f) = \mathrm{im}(f)$$

for all homomorphisms $f : G \mapsto H$, we see that

$$\mathrm{Inn}(S_3) \cong S_3/Z(S_3) \cong S_3.$$

Thus, $|\mathrm{Inn}(S_3)| = 6 = |\mathrm{Aut}(S_3)|$, and since $\mathrm{Inn}(S_3) \leq \mathrm{Aut}(S_3)$, equality holds giving

$$\mathrm{Inn}(S_3) = \mathrm{Aut}(S_3).$$

Together with $Z(S_3) = \{e\}$, this shows S_3 is complete. ∎

Lemma 3.6. *The symmetric group S_2 is not complete.*

Proof. We see that

$$S_2 = \{e, (1\,2)\}.$$

Since S_2 has only two elements, it is abelian and thus every element commutes with every other element, so

$$Z(S_2) = S_2.$$

Since the center of a complete group must be trivial, S_2 cannot be complete.
Therefore, S_2 is not complete. ∎

4. The outer automorphism of S_6

So far, the proofs we have done stem mainly from the basic properties of symmetric groups or follow pretty directly from the definition with some algebraic manipulation. On the other hand, the outer automorphism of S_6 cannot be described in any such trivial way. As a result, mathematicians have invented many ways to construct this automorphism.
The most intuitive way to solve this problem is using a visual representation of the way S_6 acts. This is why we will use combinatorial objects, such as labeled pentagons, triangles and icosahedra. This can also be done using projective space which is even more powerful, but it requires much more background knowledge and although it may be described as more algebraically rigorous it lacks the intuitive elegance that the other proofs present.

4.1. **The Six Mystic Pentagons.** [BHRD] Consider the complete graph with five vertices K_5. This graph has 10 edges and we will separate it into two disjoint cycles of length five by coloring one cycle red and the other blue. We count that there are $\frac{5!}{5} = 12$ possible cycles that cover all five of the vertices. Notice that each coloring has a complementary one so we are counting twice as many colorings as we should be. Therefore we see that we can color K_5 six ways like so which is shown in Figure 1.
Then let $P = \{P_1, P_2, ..., P_6\}$ be the set of mystic pentagons. Although the vertices of K_5 are a five element set, the colorings of the pentagons form a completely new object with six elements. Let us consider how S_5 acts on the vertex set $\{1, 2, 3, 4, 5\}$ of each pentagon. Naturally, S_5 permutes the labels of the vertices and thus also permutes the ten edges. As a result, S_5 maps every 5-cycle to another 5-cycle. Thus every element of S_5 induces a permutation of the six mystic pentagons. We can then define the group homomorphism

$$\phi : S_5 \mapsto S_6$$

where for a permutation $\sigma \in S_5$, $\phi(\sigma)$ records the rearrangement of the six mystic pentagons.

Six Mystic Pentagons.png

Figure 1. The Six Mystic Pentagons

Now suppose some σ that fixes all of the six mystic pentagons. Since each coloring is defined by its edges, σ cannot change any of the edges and thus cannot permute any of the vertices. Therefore every vertex remains fixed so

$$\sigma = e \text{ and } ker(\phi) = \{e\}.$$

Therefore ϕ is an inclusion of S_5 into S_6.
Here S_5 unusually acts on six objects which reveals a unique symmetry. For instance, consider the permutation $(1\,2) \in S_5$. This simply swaps the labels of the vertices labeled 1 and 2. This permutation sends

$$\{1, 2\} \mapsto \{1, 2\}$$

which leaves the edge $\{1, 2\}$ unchanged. On the other hand,

$$\{1,3\} \mapsto \{2,3\}$$
$$\{1,5\} \mapsto \{2,5\}$$
$$\{2,4\} \mapsto \{1,4\}$$
$$\ldots$$

After relabeling the 10 edges we obtain a new coloring of K_5 which must be one of the six mystic pentagons. Then suppose that $(1\,2)(P_1) = P_4$. Then we must have that $(1\,2)(P_4) = P_1$. We then see that $(1\,2)$ gives us a cyclic map $P_1 \leftrightarrow P_4$. Repeating this with computational verification shows that

$$P_1 \leftrightarrow P_4$$
$$P_2 \leftrightarrow P_3$$
$$P_5 \leftrightarrow P_6$$

which means the action of $(1\,2)$ on the six letters is

$$(P_1\,P_4)(P_2\,P_3)(P_5\,P_6).$$

Automorphisms of vertices of a complete graph induce automorphisms on edges, so when we color edges consistently with the color of the preimage (ie. if $\{i, j\}$ is blue, then $\{\phi(i), \phi(j)\}$ will be blue), this induces an automorphism on the set of all mystic pentagons, which is a six element set. Thus, this is an action of S_5 on S_6.
We call any cycle of length 2, like $(1\,2)$ a transposition. We notice that $(P_1\,P_4)(P_2\,P_3)(P_5\,P_6)$ is not a transposition, but a triple transposition. Therefore, because conjugation must preserve the cycle type, $\phi(S_5)$ cannot be the standard subgroup by fixing one point.
We recall that $\phi(S_5)$ is a subgroup of S_6. Then because $|S_6| = 720$ and $|S_5| = 120$, there are exactly 6 left cosets $S_6/\phi(S_5)$. It is important to note that the six mystic pentagons are distinct from these 6 left cosets and the pentagon's are only used to construct $\phi(S_5)$.
When S_6 acts on $S_6/\phi(S_5)$, we notice that this action is not isomorphic to the usual action of S_6 on a six element set. We then define the homomorphism $f : S_6 \mapsto S_6$.

Suppose that $g \in \ker(f)$. Then g fixes every left coset and

$$gx\phi(S_5) = x\phi(S_5)\ \forall\ x \in S_6.$$

We then also know that

$$x^{-1}gx \in \phi(S_5), g \in x\phi(S_5)x^{-1}\ \forall\ x \in S_6.$$

As a result

$$g \in \bigcap_{x \in S_6} x\phi(S_5)x^{-1}.$$

This means that

$$\ker(f) = \bigcap_{x \in S_6} x\phi(S_5)x^{-1}.$$

Lemma 4.1. *For $n \geq 5$ the only normal subgroups of S_n are*

$$\{e\},\ A_n,\ S_n.$$

[R]

The proof of this lemma is quite technical and lies outside of the main idea of the construction. Here, A_n denotes the set of all permutations that can be written as an even number of transpositions. Since the $\ker(f)$ is a normal subgroup of S_6, $\phi(S_5)$ must contain one of the possible normal subgroups of S_6.

$$|\phi(S_5)| = 120, \text{ while } |A_6| = 360 \text{ and } |S_6| = 720$$

, so $\phi(S_5)$ cannot contain any of these normal subgroups. Thus,

$$\ker(f) = \bigcap_{x \in S_6} x\phi(S_5)x^{-1} = \{e\}.$$

Since $\ker(f) = \{e\}$, f must be injective and because f is an injective function in between two finite groups of the same order it is also surjective and bijective.
Therefore, f is a bijective homomorphism, or an automorphism which sends

$$(1\,2) \mapsto (P_1\,P_4)(P_2\,P_3)(P_5\,P_6).$$

An inner automorphism must send one cycle type to the same cycle type, but we can clearly see that this is not the case here as we are sending transpositions to triple transpositions. Therefore f is an outer automorphism of S_6 and thus S_6 is not complete.

4.2. **Labeled Triangles.** [BHRD]
We now make the construction more symmetric by considering the set of all 3-element subsets of $\{1,\ldots,6\}$, which can be denoted as

$$\binom{\{1,\ldots,6\}}{3},$$

which consists of 20 triangles. This is simply the ways to choose 3 elements from [6]. We think of these triangles as the faces of the complete 3-uniform hypergraph on six vertices. Two triangles are *disjoint* if they share no vertices, and a *tetrahedron* refers to a choice of 4 vertices, which determines exactly 4 triangles.

We now seek a two-coloring (red/blue) of the 20 triangles satisfying:

(i) Disjoint triangles have opposite colors,
(ii) Every tetrahedron contains exactly two red and two blue triangles.

Condition (i) is extremely rigid: if two triangles do not share vertices, then knowing the color of one forces the color of the other. In particular, once we choose the color of a single triangle, the colors of all triangles disjoint from it are forced.
Condition (ii) prevents degeneracy: it ensures that the coloring is globally balanced on every 4-vertex subsystem, so the rule from (i) does not collapse into a contradiction when propagated around cycles of overlapping tetrahedra.

Now we must count the possible colorings: Fix a triangle, say $\{1,2,3\}$. Without loss of generality we may assume it is colored red.
Any triangle disjoint from $\{1,2,3\}$ must then be colored blue by condition (i). The only triangle disjoint from $\{1,2,3\}$ is $\{4,5,6\}$, so this triangle is forced to be blue.
Now consider triangles sharing exactly one vertex with $\{1,2,3\}$. These lie in overlapping tetrahedra, and condition (ii) forces each tetrahedron containing $\{1,2,3\}$, exactly two of the four triangles must be red. This forces a rigid pairing behavior among triangles sharing two vertices with $\{1,2,3\}$.

Carrying this propagation through all tetrahedra shows that once one triangle is chosen, the entire coloring is determined uniquely.

We now construct six distinct colorings explicitly.
We now modify the mystic pentagon construction so that no particular element of $\{1, \ldots, 6\}$ is distinguished. For each $(i \in 1, \ldots, 6)$, remove the vertex (i). The remaining five vertices form a complete graph K_5, on which we choose one of the six mystic pentagon colorings. This coloring determines a coloring of all 20 triangles on the original six vertices. If a triangle contains the vertex i, say $\{i, a, b\}$, then it is assigned the same color as the edge $\{a, b\}$ in the corresponding mystic pentagon on K_5. If a triangle does not contain i, say $\{a, b, c\}$, then the remaining two vertices form the complementary edge in K_5, and the triangle is assigned the opposite color of this edge. Repeating this construction for each choice of (i) produces six distinct colorings $T_1, \ldots, T_6$ of the 20 triangles. These colorings satisfy two properties: any two disjoint triangles receive opposite colors, and every tetrahedron contains exactly two triangles of each color. Since every permutation of the six vertices preserves incidences between vertices, edges, and triangles, each element of S_6 permutes the six colorings $T_1, \ldots, T_6$.
It is straightforward to verify that each coloring T_i satisfies conditions (i) and (ii). Conversely, if a coloring of the 20 triangles satisfies conditions (i) and (ii), then it is equal to T_i for a unique $i \in \{1, \ldots, 6\}$. Therefore, the colorings $T_1, \ldots, T_6$ are precisely the six colorings satisfying conditions (i) and (ii) and there are at least six valid colorings.

We now show that these are the only possible colorings.
Let C be any coloring of the 20 triangles satisfying conditions (i) and (ii). Fix a vertex $i \in \{1, \ldots, 6\}$. Restrict C to the triangles not containing i. These triangles correspond to the 10 edges of the complete graph K_5 on the remaining vertices $\{1, \ldots, 6\} \setminus \{i\}$. Under this correspondence, the induced edge coloring has the property that each color class forms a 5-cycle. Indeed, conditions (i) and (ii) translate precisely into the requirement that the red and blue edges each form a cycle in K_5. Therefore, by the classification proved earlier, this induced coloring must be one of the six mystic pentagon colorings.
Thus C is uniquely determined by the induced pentagon structure on the complement of a vertex, and hence must equal one of the T_i.
Therefore there are exactly six valid colorings.
Let

$$T = \{T_1, T_2, \ldots, T_6\}$$

be the set of these six colorings.
We now describe the key symmetry behind the construction. The group S_6 acts naturally on the set of vertices $\{1, \ldots, 6\}$, and this induces an action on the set T of all 20 triangles. Hence, if

$$\sigma \in S_6$$

and a triangle is represented by the three-element subset

$$\{i, j, k\} \subseteq \{1, \ldots, 6\},$$

then σ sends this triangle to the triangle

$$\sigma(\{i, j, k\}) = \{\sigma(i), \sigma(j), \sigma(k)\}.$$

In other words, σ simply relabels the vertices of each triangle according to the permutation σ.

This action preserves the combinatorial structure of the six vertices: disjoint triangles are sent to disjoint triangles, and sets of four mutually adjacent triangles forming tetrahedra are sent to sets of triangles forming tetrahedra. Therefore, the action preserves the defining properties of a valid coloring. Hence every $\sigma \in S_6$ induces a permutation of the set of valid colorings, and in particular gives a permutation of the six mystic pentagons. This defines a homomorphism

$$\psi : S_6 \longrightarrow S_6.$$

We claim that ψ is injective. Suppose $\sigma \in \ker(\psi)$. Then σ fixes every triangle coloring, and preserves the induced partition of all 20 triangles. In particular, σ preserves all incidence relations among vertices, and therefore fixes every 3-subset of $\{1, \dots, 6\}$. It follows that σ fixes every pair of vertices, hence every element of $\{1, \dots, 6\}$. Thus $\sigma = e$, so $\ker(\psi) = \{e\}$. Hence ψ is injective. Since both domain and codomain have order 720, it follows that ψ is an isomorphism.

To see that this automorphism is not inner, we consider the image of a transposition once again. Consider the transposition $(1\,2) \in S_6$. Under the construction above, $(1\,2)$ acts by relabeling the vertices of each mystic pentagon. Applying this relabeling to each of the six valid colorings, we obtain the permutation induced on the set

$$\{T_1, \dots, T_6\}.$$

A direct computation shows that

$$\psi((1\,2)) = (T_1\,T_4)(T_2\,T_3)(T_5\,T_6),$$

for our first labeling of the six mystic pentagons. Thus the image of a transposition in S_6 is a triple transposition.
Therefore ψ cannot be an inner automorphism, and the action of S_6 on the six colorings realizes the outer automorphism of S_6.

4.3. **Labeled Icosahedra.** [BHRD]

We will now look at another symmetric realization of the same phenomenon, which uses the geometry of the icosahedron.
Consider a regular icosahedron. We label its vertices by the set $\{1, \dots, 6\}$ in such a way that antipodal vertices receive the same label. Since the icosahedron has 12 vertices arranged into 6 antipodal pairs, this labeling identifies each pair of opposite vertices with a single label.
The 20 triangular faces of the icosahedron occur in 10 pairs of opposite faces. Each pair of opposite triangular faces has the same three vertex labels, and therefore determines an unordered triple of labels. Hence such a labeling gives rise to 10 unordered triples of labels, corresponding to the 10 pairs of antipodal triangular faces.
We will study the different ways of assigning labels to the antipodal vertex pairs and the resulting collections of triples. These labelings encode the combinatorial structure of the icosahedron and will allow us to relate its symmetries to permutations of the six labels.
Thus each labeled icosahedron encodes a distinguished structure on the set

$$\binom{\{1, \dots, 6\}}{3}.$$

Unlike the triangle construction, which begins with 20 combinatorial objects, the icosahedron organizes these objects geometrically. Once the labels of the three antipodal vertex pairs appearing in a single face are fixed, the remaining labels are constrained by the adjacency

relations of the icosahedron. Moving from one triangular face to an adjacent face determines how the labels must be assigned to the neighboring vertices, and this process continues throughout the entire solid.
The resulting labeling is therefore determined uniquely up to the symmetries of the icosahedron. Thus, after fixing an initial face, the remaining structure is forced except for the natural rotational and reflectional symmetries of the icosahedron.

A label is one of the symbols in the set $\{1, \ldots, 6\}$, while a labeling is a complete assignment of these six labels to the icosahedron. Since a regular icosahedron has 12 vertices arranged into 6 pairs of opposite vertices, we assign the same label to the two vertices in each antipodal pair. Thus, a labeling is equivalent to choosing a one-to-one correspondence between the six antipodal vertex pairs and the six labels $\{1, \ldots, 6\}$.
Equivalently, we choose which antipodal vertex pair receives label 1, which receives label 2, and so on. Different choices of this correspondence give different labelings of the icosahedron. Up to the symmetries of the icosahedron (rotations and reflections), there are exactly 12 such labelings.
We now organize these 12 labelings into six pairs. Two labelings are called *opposite* if they share no unordered triple of labels arising from a pair of opposite triangular faces. Let

$$I = \{I_1, I_2, \ldots, I_6\}$$

denote the set of these six opposite pairs.

The symmetric group S_6 acts naturally on $\{1, \ldots, 6\}$, hence on vertex labelings of the icosahedron. This action preserves incidence relations between vertices and faces, and therefore sends labeled icosahedra to labeled icosahedra.
In addition, opposite pairs are preserved under this action, so each $\sigma \in S_6$ induces a permutation of the six-element set I. This defines a homomorphism

$$\theta : S_6 \longrightarrow S_6.$$

Suppose $\sigma \in \ker(\theta)$. Then σ fixes every opposite pair of icosahedra, hence fixes all induced collections of face-triples.
Since every pair of vertices appears in some face-triple, this forces σ to preserve all 3-subsets of $\{1, \ldots, 6\}$. In particular, σ preserves incidence between pairs of vertices and triples.
It follows that σ must fix each element of $\{1, \ldots, 6\}$, hence $\sigma = e$. Therefore,

$$\ker(\theta) = \{e\},$$

so θ is injective.
Since both domain and codomain have order 720, it follows that θ is an isomorphism.

Consider a transposition $(1\,2) \in S_6$. Its induced action on the six pairs of icosahedra does not correspond to a single transposition of the set I, but instead as we have seen in the previous two constructions, this transposition is sent to a triple transposition.
Hence θ cannot arise from conjugation in S_6, and is therefore an outer automorphism.

Heuristic 4.2. ***Intuitive picture of the projective construction of S_6.***
The connection between finite groups and geometric configurations has a long history in projective geometry [K]. One way to understand the outer automorphism of S_6 is to reinterpret

the six objects being permuted as geometric points rather than labels on a combinatorial or geometric object.
Consider six points in the projective plane

$$\{p_1, p_2, p_3, p_4, p_5, p_6\} \subset \mathbb{P}^2.$$

Here $\mathbb{P}^2$ *is the space of lines through the origin in a* 3*-dimensional vector space. The group of projective transformations is*

$$\mathrm{PGL}(3) = \mathrm{GL}(3)/\{\textit{nonzero scalar transformations}\},$$

since multiplying a linear transformation by a scalar does not change the corresponding projective transformation. These transformations move the six points while preserving the projective structure.
Therefore, rather than studying individual configurations, we study configurations up to projective equivalence:

$$(\mathbb{P}^2)^6 / \mathrm{PGL}(3).$$

The symmetric group S_6 *acts naturally on this space by permuting the six points:*

$$\sigma(p_1, p_2, p_3, p_4, p_5, p_6) = (p_{\sigma(1)}, p_{\sigma(2)}, \dots, p_{\sigma(6)}).$$

The remarkable feature is that this geometric space contains symmetries that are not explained by simply relabeling the six points. These additional symmetries interchange different geometric descriptions of the same configuration and give rise to an automorphism of S_6 *that cannot be obtained by conjugation. In this way, projective geometry provides a geometric explanation for the outer automorphism of* S_6*.*

5. S_n is complete for $n \neq 2, 6$

To prove **Theorem 3.2** we must prove three different statements:

1. The center of S_n is trivial for $n \geq 3$, $Z(S_n) = \{e\}$.
2. Any automorphism preserving transpositions is conjugation by a permutation.
3. Every automorphism of S_n sends transpositions to transpositions for $n \neq 6$.

Note that S_1 is trivially simple as it is defined as the trivial group with only an identity.
The first two statements are quite simple to prove and follow directly from definitions, while the third is quite involved.

Lemma 5.1. *For* $n \geq 3$*, the center of* S_n *is trivial. That is,*

$$Z(S_n) = \{e\} \text{ [S].}$$

Proof. Suppose $\sigma \in Z(S_n)$. We will show that $\sigma = e$.

For the sake of contradiction, assume that $\sigma \neq e$. Then there exists some $i \in \{1, \dots, n\}$ such that

$$\sigma(i) = j \neq i.$$

Since $n \geq 3$, we can choose an element

$$k \neq i, j.$$

Then, consider the transposition

$$\tau = (i\,k).$$

Since σ is central,

$$\sigma\tau = \tau\sigma.$$

Evaluate both sides at i. We obtain

$$\sigma\tau(i) = \sigma(k),$$

while

$$\tau\sigma(i) = \tau(j) = j,$$

since $j \neq i, k$.
As a result, we see that

$$\sigma(k) = j.$$

However, $\sigma(i) = j$ as well, contradicting the injectivity of σ.
Therefore no nonidentity permutation lies in the center, and

$$Z(S_n) = \{e\}.$$

■

Lemma 5.2. *Let $\phi \in \mathrm{Aut}(S_n)$, where $n \neq 6$. If ϕ sends transpositions to transpositions, then ϕ is an inner automorphism.*

Proof. For each $i \in \{1, \dots, n\}$, let

$$T_i = \{(i\,j) : j \neq i\},$$

be the set of all transpositions containing the symbol i.
Observe that any two distinct transpositions in T_i do not commute, since they share the element i. Conversely, if two transpositions do not commute, then they share exactly one symbol. Therefore, for each i, the set T_i can be characterized purely in terms of the group operation: it is a maximal collection of transpositions that are pairwise non-commuting.
Since this characterization depends only on the multiplication structure of S_n, it is preserved by automorphisms. Hence any automorphism of S_n must send the collection

$$\{T_1, T_2, \dots, T_n\}$$

to itself, possibly permuting the sets among one another. In this way, the group structure itself recovers the natural action of S_n on the underlying set $\{1, \dots, n\}$.
Since ϕ preserves transpositions and preserves commutativity, it permutes the sets

$$T_1, T_2, \dots, T_n.$$

Therefore there exists a permutation

$$\pi \in S_n$$

such that

$$\phi(T_i) = T_{\pi(i)}$$

for every i.
Then consider a transposition $(i\,j)$. It is the unique element contained in both T_i and T_j. Therefore,

$$\phi((i\,j))$$

is the unique transposition lying in both, which for some permutation π

$$T_{\pi(i)} \quad \text{and} \quad T_{\pi(j)},$$

which is equivalent to

$$(\pi(i)\,\pi(j)).$$

Thus,

$$\phi((i\,j)) = (\pi(i)\,\pi(j)) = \pi(i\,j)\pi^{-1},$$

where the last equality follows from the fact that conjugation sends

$$(a_1\,a_2\,\cdots\,a_k)$$

to

$$(\pi(a_1)\,\pi(a_2)\,\cdots\,\pi(a_k)).$$

Since the transpositions generate S_n, every permutation

$$\sigma \in S_n$$

can be written as a product of transpositions,

$$\sigma = t_1 t_2 \cdots t_r.$$

Using that ϕ is a homomorphism, we obtain

$$\begin{aligned}
\phi(\sigma) &= \phi(t_1 t_2 \cdots t_r) \\
&= \phi(t_1)\phi(t_2)\cdots\phi(t_r) \\
&= (\pi t_1 \pi^{-1})(\pi t_2 \pi^{-1})\cdots(\pi t_r \pi^{-1}) \\
&= \pi t_1 t_2 \cdots t_r \pi^{-1} \\
&= \pi\sigma\pi^{-1}.
\end{aligned}$$

Therefore,

$$\phi(\sigma) = \pi\sigma\pi^{-1}$$

for every $\sigma \in S_n$. Hence ϕ is conjugation by the permutation π, so ϕ is an inner automorphism.

■

Definition 5.3. An element g of a group G is called an *involution* if

$$g \neq e$$

and

$$g^2 = e,$$

where e is the identity element of G.

Definition 5.4. Let G be a group and let $g \in G$. The *centralizer* of g in G is the set

$$C_G(g) = \{x \in G : xg = gx\}.$$

That is, the centralizer of g consists of all elements of G that commute with g.

Lemma 5.5. *Let G be a group, let $\phi \in \mathrm{Aut}(G)$, and let $g \in G$. Then*

$$\phi(C_G(g)) = C_G(\phi(g)).$$

As a result,

$$|C_G(g)| = |C_G(\phi(g))|.$$

In other words, automorphisms preserve centralizers.

Proof. Let $x \in C_G(g)$. Then
$$xg = gx.$$
Since ϕ is a homomorphism,
$$\phi(xg) = \phi(gx),$$
which implies
$$\phi(x)\phi(g) = \phi(g)\phi(x).$$
Thus
$$\phi(x) \in C_G(\phi(g)).$$
Hence,
$$\phi(C_G(g)) \subseteq C_G(\phi(g)).$$
To prove the reverse inclusion, let $y \in C_G(\phi(g))$. Then
$$y\phi(g) = \phi(g)y.$$
Since ϕ is an automorphism, it is bijective, so there exists $x \in G$ such that
$$y = \phi(x).$$
Applying ϕ^{-1} to the equation
$$\phi(x)\phi(g) = \phi(g)\phi(x)$$
gives
$$xg = gx.$$
Hence
$$x \in C_G(g),$$
and therefore
$$y = \phi(x) \in \phi(C_G(g)).$$
Thus,
$$C_G(\phi(g)) \subseteq \phi(C_G(g)).$$
Combining the two inclusions yields
$$\phi(C_G(g)) = C_G(\phi(g)).$$
Since ϕ is a bijection,
$$|C_G(g)| = |\phi(C_G(g))| = |C_G(\phi(g))|,$$
as desired. ∎

Lemma 5.6. *Let $\phi \in \mathrm{Aut}(S_n)$, where $n \neq 6$. Then ϕ sends transpositions to transpositions.*

Proof. Automorphisms preserve the order of elements, so ϕ sends involutions to involutions. Every involution in S_n is conjugate to a permutation of the form
$$(1\,2)(3\,4)\cdots(2k-1\ 2k)$$
for some integer $k \geq 1$. Thus, if
$$t = (1\,2),$$
then $\phi(t)$ must be conjugate to a product of k disjoint transpositions for some k. Our goal is to determine which values of k are possible.

To do this, we use the fact that automorphisms preserve centralizers. Intuitively, the centralizer of an element measures how many permutations commute with it. Since automorphisms

preserve the group operation, they also preserve commuting relationships. Consequently, an automorphism cannot change the size of an element's centralizer.
We first compute the centralizer of the transposition

$$t = (1\,2).$$

A permutation commutes with $(1\,2)$ precisely when it preserves the set

$$\{1, 2\}.$$

Notice that if a permutation were to move exactly one of 1 or 2 outside this set, then conjugating by that permutation would change the transposition. Thus, a permutation in the centralizer must either fix both elements or interchange them, while acting arbitrarily on the remaining $n-2$ elements. Therefore,

$$|C_{S_n}(t)| = 2(n-2)!.$$

Now consider the involution

$$x = (1\,2)(3\,4)\cdots(2k-1\ 2k).$$

To commute with x, a permutation must preserve the collection of transposition pairs. It may permute these k pairs in any order, independently swap the two elements within each pair, and act arbitrarily on the remaining $n-2k$ elements. Since there are

$$k!$$

ways to permute the pairs,

$$2^k$$

ways to choose which pairs are swapped, and

$$(n-2k)!$$

ways to permute the remaining elements, we obtain

$$|C_{S_n}(x)| = 2^k k!(n-2k)!.$$

Since automorphisms preserve centralizers,

$$|C_{S_n}(t)| = |C_{S_n}(\phi(t))|,$$

and therefore

$$2(n-2)! = 2^k k!(n-2k)!.$$

Cancelling the common factor $2(n-2k)!$ gives

$$(n-2)(n-3)\cdots(n-2k+1) = 2^{k-1}k!.$$

We now determine the possible values of k.
If $k = 1$, the equality clearly holds.
If $k = 2$, the equation becomes

$$(n-2)(n-3) = 4,$$

which has no integer solution.
If $k = 3$, we obtain

$$(n-2)(n-3)(n-4)(n-5) = 24.$$

The unique integer solution is

$$n = 6.$$

Finally, suppose $k \geq 4$. The left-hand side is a polynomial in n that is strictly increasing for all $n \geq 2k$, so its smallest possible value occurs when $n = 2k$. In that case it equals

$$(2k-2)!.$$

Since

$$(2k-2)! > 2^{k-1}k!$$

for every $k \geq 4$, the equation cannot hold. Thus there are no solutions when $k \geq 4$.
We conclude that, whenever $n \neq 6$, the only possible value is

$$k = 1.$$

Hence $\phi(t)$ is itself a transposition.
Finally, every transposition in S_n is conjugate to $(1\,2)$, and automorphisms preserve conjugacy classes. Therefore every transposition is mapped to another transposition.

Hence every automorphism of S_n sends transpositions to transpositions whenever $n \neq 6$. ■

Combining these three statements we see that the center of every S_n is trivial for $n \geq 3$. And we know that all automorphisms that send transpositions to transpositions are inner, and every automorphism of S_n sends transpositions to transpositions for $n \neq 6$. Therefore, S_n is complete for $n \neq 2, 6$. Although, this proves that S_n does not have any outer automorphisms for $n \neq 6$, it doesn't prove that S_6 has an outer automorphism. This is the reason we still study and develop explicit constructions of this unique outer automorphism.

6. Conclusion

Although the outer automorphism of S_6 is a result which is important due to its unique nature and beauty, symmetric groups play an important role in many other places in mathematics. They arise naturally whenever one studies permutations, symmetries, or finite group actions, and they appear in subjects ranging from Galois theory and combinatorics to geometry, topology, and mathematical physics. Complete groups are equally important because their symmetries are entirely determined by the group itself. In a complete group, every automorphism comes from conjugation, so there are no unexpected symmetries. This rigidity makes complete groups fundamental objects in the classification and study of finite groups.

The symmetric group S_6 is the unique exception among the symmetric groups. Rather than being complete, it possesses a single nontrivial outer automorphism that exchanges transpositions with triple transpositions. Although this symmetry cannot be detected from the usual action of permutations on six symbols, it appears naturally through several remarkable constructions involving mystic pentagons, labeled triangles, labeled icosahedra, and projective geometry. Each construction reveals the same hidden symmetry from a different perspective.

In this paper, we introduced the necessary background on groups, automorphisms, and complete groups before proving that S_n is complete for every $n \neq 2, 6$. We then constructed the exceptional outer automorphism of S_6 using several combinatorial and geometric models, showing that all of these different constructions describe the same unique phenomenon. The outer automorphism of S_6 remains one of the most beautiful and surprising exceptions in finite group theory, demonstrating how an isolated algebraic curiosity can reveal unexpected complexity that can be explored through many mathematical lenses.

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